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Mathematical Analysis

A new characterization of Sobolev spaces

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Abstract

Our main result is the following: Let $g \in L^p(\mathbb{R}^N)$, $1 < p < +\infty$, be such that

$$\sup_{n\in\mathbb{N}} \int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N \\ |g(x)-g(y)|>\delta_n}} \frac{\delta_n^p}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y < +\infty,$$

for some arbitrary sequence of positive numbers $(\delta_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}\delta_n = 0$. Then $g \in W^{1,p}(\mathbb{R}^N)$.

This extends a result from H.-M. Nguyen (2006). ***To cite this article: J. Bourgain, H.-M. Nguyen, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Une nouvelle caractérisation des espaces de Sobolev. Notre résultat principal est le suivant : Soit une fonction $g \in L^p(\mathbb{R}^N)$, $1 < p < +\infty$, telle que

$$\sup_{n\in\mathbb{N}} \int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N \\ |g(x)-g(y)|>\delta_n}} \frac{\delta_n^p}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y < +\infty,$$

où $(\delta_n)_{n\in\mathbb{N}}$ est une suite arbitraire positive telle que $\lim_{n\to\infty}\delta_n = 0$. Alors $g \in W^{1,p}(\mathbb{R}^N)$.

Cela étend un résultat de H.-M. Nguyen (2006). ***Pour citer cet article : J. Bourgain, H.-M. Nguyen, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

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Notre résultat principal est le Théorème 1. Il étend un résultat dans [5] où la même conclusion a été obtenue avec l'hypothèse plus forte :

$$\sup_{0<\delta<1} \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\limits_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y < +\infty.$$

L'argument dans [5] a utilisé une intégration de (2) par rapport à δ et ne serait pas adapté à la condition (1). La méthode qu'on utilise ici est tout à fait différente et beaucoup plus délicate.

1. Introduction

Our main result is the following:

Theorem 1. *Let $g \in L^p(\mathbb{R}^N)$, $1 < p < +\infty$, be such that*

$$\sup_{n\in\mathbb{N}} \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\limits_{|g(x)-g(y)|>\delta_n} \frac{\delta_n^p}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y < +\infty, \tag{1}$$

for some sequence of positive numbers $(\delta_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}\delta_n = 0$. Then $g \in W^{1,p}(\mathbb{R}^N)$.

This extends a result from [5]. In [5] the second author obtained the same conclusion under the stronger assumption that

$$\sup_{0<\delta<1} \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\limits_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y < +\infty. \tag{2}$$

The argument in [5] used an integration of (2) with respect to δ and could not be adapted to the assumption (1). The method we present here is totally different and much more delicate.

2. Proof of Theorem 1

The following lemma is the main ingredient in the proof of Theorem 1.

Lemma 2. *Let f be a measurable function on a bounded nonempty interval I and $1 \leqslant p < +\infty$. Then*

$$\liminf_{\varepsilon\to 0_+} \iint\limits_{I\times I}\limits_{|f(x)-f(y)|>\varepsilon} \frac{\varepsilon^p}{|x-y|^{p+1}}\,\mathrm{d}x\,\mathrm{d}y \geqslant c\frac{1}{|I|^{p-1}}\Big(\operatorname*{ess\,sup}_I f - \operatorname*{ess\,inf}_I f\Big)^p, \tag{3}$$

where $c = c_p$ is a positive constant depending only on p.

Proof of Lemma 2. *Step 1.* (3) holds if $f \in L^\infty(I)$.

By rescaling, we may assume $I = [0, 1]$.

Denote $s_+ = \operatorname{ess\,sup}_I f$, $s_- = \operatorname{ess\,inf}_I f$. Rescaling f, one may also assume

$$s_+ - s_- = 1 \tag{4}$$

(unless f is constant on I in which case there is nothing to prove).

Take $0 < \delta \ll 1$ small enough to ensure that there are (density) points $t_+, t_- \in [20\delta, 1 - 20\delta] \subset [0, 1]$ with

$$\begin{cases} \left|[t_+ - \tau, t_+ + \tau] \cap \left[f > \frac{3}{4}s_+ + \frac{1}{4}s_-\right]\right| > \frac{9}{5}\tau, \\ \left|[t_- - \tau, t_- + \tau] \cap \left[f < \frac{3}{4}s_- + \frac{1}{4}s_+\right]\right| > \frac{9}{5}\tau, \end{cases} \quad \forall 0 < \tau < 20\delta. \tag{5}$$

Take $K \in \mathbb{Z}_+$ such that $\delta < 2^{-K} \leqslant 5\delta/4$ and denote

$$J = \left\{ j \in \mathbb{Z}_+;\ \frac{3}{4}s_- + \frac{1}{4}s_+ < j2^{-K} < \frac{3}{4}s_+ + \frac{1}{4}s_- \right\}.$$

Then

$$|J| \geqslant 2^{K-1} - 2 \approx \frac{1}{\delta}. \tag{6}$$

For each j, define the following sets:

$$A_j = \left\{x \in [0, 1];\ (j-1)2^{-K} \leqslant f(x) < j2^{-K}\right\}, \qquad B_j = \bigcup_{j'<j} A_{j'} \quad \text{and} \quad C_j = \bigcup_{j'>j} A_{j'},$$

so that $B_j \times C_j \subset [|f(x) - f(y)| \geqslant 2^{-K}] \subset [|f(x) - f(y)| > \delta]$.

Since the sets A_j are disjoint, it follows from (6) that

$$\operatorname{card}(G) \geqslant 2^{K-2} - 3 \approx \frac{1}{\delta}, \tag{7}$$

where G is defined by

$$G = \left\{ j \in J;\ |A_j| < 2^{-K+2} \right\}.$$

For each $j \in J$, set $\lambda_{1,j} = |A_j|$ and consider the function $\psi_1(t)$ defined as follows:

$$\psi_1(t) = \left|[t - 4\lambda_{1,j}, t + 4\lambda_{1,j}] \cap B_j\right|, \quad \forall t \in [20\delta, 1 - 20\delta].$$

Then, from (5), $\psi_1(t_+) < 4\lambda_{1,j}$ and $\psi_1(t_-) > 4\lambda_{1,j}$. Hence, since ψ_1 is a continuous function on the interval $[20\delta, 1 - 20\delta]$ containing the two points t_+ and t_-, there exists $t_{1,j} \in [20\delta, 1 - 20\delta]$ such that

$$\psi_1(t_{1,j}) = 4\lambda_{1,j}. \tag{8}$$

Since $\iint_{\substack{I \times I \\ |f(x)-f(y)|>\delta}} \frac{1}{|x-y|^{p+1}}\,\mathrm{d}x\,\mathrm{d}y < +\infty$, it follows that $|[t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}] \cap A_j| > 0$.

In fact, suppose $|[t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}] \cap A_j| = 0$. Then

$$\iint_{\substack{x \in [t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}] \cap B_j \\ y \in [t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}] \setminus B_j}} \frac{1}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y \leqslant \iint_{\substack{I \times I \\ |f(x)-f(y)|>\delta}} \frac{1}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y < +\infty.$$

Hence $|[t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}] \cap B_j| = 0$ or $|[t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}] \setminus B_j| = 0$ (see [3]). This is a contradiction since $\psi_1(t_{1,j}) = |[t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}] \cap B_j| = 4\lambda_{1,j}$ (see (8)).

If $|[t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}] \cap A_j| < \lambda_{1,j}/4$, then take $\lambda_{2,j} > 0$ such that $\lambda_{1,j}/\lambda_{2,j} \in \mathbb{Z}_+$ and

$$\frac{\lambda_{2,j}}{2} < \left|[t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}] \cap A_j\right| \leqslant \lambda_{2,j}.$$

Since $|[t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}] \cap A_j| < \lambda_{1,j}/4$, we infer that $\lambda_{2,j} \leqslant \lambda_{1,j}/2$.

Set $E_{2,j} = [t_{1,j} - 4\lambda_{1,j} + 4\lambda_{2,j}, t_{1,j} + 4\lambda_{1,j} - 4\lambda_{2,j}]$ and consider the function $\psi_2(t)$ defined as follows

$$\psi_2(t) = \left|[t - 4\lambda_{2,j}, t + 4\lambda_{2,j}] \cap B_j\right|, \quad \forall t \in E_{2,j}.$$

We claim that there exists $t_{2,j} \in E_{2,j}$ such that $\psi_2(t_{2,j}) = 4\lambda_{2,j}$.

To see this, we argue by contradiction. Suppose that $\psi_2(t) \neq 4\lambda_{2,j}$, for all $t \in E_{2,j}$. Since ψ_2 is a continuous function on $E_{2,j}$, we assume as well that $\psi_2(t) < 4\lambda_{2,j}$, for all $t \in E_{2,j}$. Since $\lambda_{1,j}/\lambda_{2,j} \in \mathbb{Z}_+$, it follows that $\psi_1(t_{1,j}) < 4\lambda_{1,j}$, hence we have a contradiction to (8).

It is clear that

$$\iint\limits_{\substack{[t_{2,j}-4\lambda_{2,j},t_{2,j}+4\lambda_{2,j}]^2\\ |f(x)-f(y)|>\delta}} \frac{1}{|x-y|^{p+1}}\,\mathrm{d}x\,\mathrm{d}y \geqslant \iint\limits_{\substack{[t_{2,j}-4\lambda_{2,j},t_{2,j}+4\lambda_{2,j}]^2\\ x\in B_j;\, y\in C_j}} \frac{1}{|x-y|^{p+1}}\,\mathrm{d}x\,\mathrm{d}y \gtrsim 1.$$

If $|[t_{2,j}-4\lambda_{2,j}, t_{2,j}+4\lambda_{2,j}] \cap A_j| < \lambda_{2,j}/4$, then take $\lambda_{3,j}$ ($\lambda_{3,j} \leqslant \lambda_{2,j}/2$) and $t_{3,j}$, etc.

On the other hand, since $\iint_{\substack{I\times I\\ |f(x)-f(y)|>\delta}} \frac{1}{|x-y|^{p+1}}\,\mathrm{d}x\,\mathrm{d}y < +\infty$, we have

$$\limsup_{\substack{\mathrm{diam}(Q)\to 0\\ Q:\text{ an interval of } I}} \iint\limits_{\substack{Q\times Q\\ |f(x)-f(y)|>\delta}} \frac{1}{|x-y|^{p+1}}\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{9}$$

Thus, from (9) and the construction of $t_{k,j}$ and $\lambda_{k,j}$, there exist $t_j \in [20\delta, 1-20\delta]$ and $\lambda_j > 0$ ($t_j = t_{k,j}$ and $\lambda_j = \lambda_{k,j}$ for some k) such that

$$\text{(a) } \big|[t_j-4\lambda_j, t_j+4\lambda_j] \cap B_j\big| = 4\lambda_j \quad \text{and} \quad \text{(b) } \frac{\lambda_j}{4} \leqslant \big|[t_j-4\lambda_j, t_j+4\lambda_j] \cap A_j\big| \leqslant \lambda_j. \tag{10}$$

Set $\lambda = \inf_{j\in G} \lambda_j$ ($\lambda > 0$ since G is finite). Suppose $G = \bigcup_{i=1}^n I_m$, where I_m is defined as follows

$$I_m = \big\{ j \in G;\ 2^{m-1}\lambda \leqslant \lambda_j < 2^m\lambda \big\}, \quad \forall m \geqslant 1.$$

Then it follows from (7) that

$$\sum_{m=1}^n \mathrm{card}(I_m) \gtrsim \frac{1}{\delta}. \tag{11}$$

For each m ($1 \leqslant m \leqslant n$), since $A_j \cap A_k = \emptyset$ for $j \neq k$, it follows from (10-b) that there exists $J_m \subset I_m$ such that

$$\text{(a) } \mathrm{card}(J_m) \gtrsim \mathrm{card}(I_m) \quad \text{and} \quad \text{(b) } |t_i - t_j| > 2^{m+3}\lambda, \quad \forall i, j \in J_m. \tag{12}$$

Then, from (12-b) and the definition of I_m,

$$[t_i-4\lambda_i, t_i+4\lambda_i] \cap [t_j-4\lambda_j, t_j+4\lambda_j] = \emptyset, \quad \forall i, j \in J_m. \tag{13}$$

Set $U_0 := \emptyset$ and

$$\begin{cases} L_m = \big\{ j \in J_m;\ \big|[t_j-4\lambda_j, t_j+4\lambda_j] \setminus U_{m-1}\big| \geqslant 6\lambda_j \big\},\\ U_m = \Big(\bigcup\limits_{j\in L_m} [t_j-4\lambda_j, t_j+4\lambda_j] \Big) \cup U_{m-1}, & \text{for } m = 1, 2, \ldots, n.\\ a_m = \mathrm{card}(J_m) \quad \text{and} \quad b_m = \mathrm{card}(L_m), \end{cases}$$

From (13) and the definitions of J_m and L_m,

$$\frac{1}{4} 2^{m-1}(a_m - b_m) \leqslant \sum_{i=1}^{m-1} 2^i b_i$$

which shows that

$$a_m \leqslant b_m + 8 \sum_{i=1}^{m-1} 2^{(i-m)} b_i.$$

Consequently,

$$\sum_{m=1}^n a_m \leqslant \sum_{m=1}^n b_m + 8 \sum_{m=1}^n \sum_{i=1}^{m-1} 2^{(i-m)} b_i = \sum_{m=1}^n b_m + 8 \sum_{i=1}^n b_i \sum_{m=i+1}^n 2^{(i-m)}.$$

Since $\sum_{i=1}^{\infty} 2^{-i} = 1$, it follows from (11) and (12-a) that

$$\sum_{m=1}^{n} b_m \geqslant \frac{1}{9} \sum_{m=1}^{n} a_m \gtrsim \frac{1}{\delta}.$$

Therefore, it is easy to see that

$$\iint\limits_{\substack{I\times I\\ |f(x)-f(y)|>\delta}} \frac{1}{|x-y|^{p+1}}\,\mathrm{d}x\,\mathrm{d}y \geqslant \sum_{m=1}^{n}\sum_{j\in L_m} \iint\limits_{\substack{([t_j-4\lambda_j,t_j+4\lambda_j]\setminus U_{m-1})^2\\ x\in B_j,\,y\in C_j}} \frac{1}{|x-y|^{p+1}}\,\mathrm{d}x\,\mathrm{d}y \gtrsim \sum_{m=1}^{n} \frac{b_m}{\delta^{p-1}} \gtrsim \frac{1}{\delta^p},$$

which yields the conclusion of Lemma 2.

Step 2. Proof of Lemma 2 completed.

Observe that if we define the function

$$f_A = \big(f \vee (-A)\big) \wedge A,$$

then

$$\big|f_A(x) - f_A(y)\big| \leqslant \big|f(x) - f(y)\big|.$$

Applying (3) to the sequence f_A and letting A goes to infinity, we deduce that (3) holds for any measurable function f on I (allowing the right-hand side to be $+\infty$). □

Proof of Theorem 1 when N = 1. Set $\tau_h(g)(x) = \frac{g(x+h)-g(x)}{h}$, $\forall x \in \mathbb{R}$, $0 < h < 1$.

For each $m \geqslant 2$, take $K \in \mathbb{R}_+$ such that $m \leqslant Kh$, then

$$\int_{-m}^{m} \big|\tau_h(g)(x)\big|^p\,\mathrm{d}x \leqslant \sum_{k=-K}^{K} \int_{kh}^{(k+1)h} \big|\tau_h(g)(x)\big|^p\,\mathrm{d}x.$$

Thus, since

$$\int_{a}^{a+h} \big|\tau_h(g)(x)\big|^p\,\mathrm{d}x \leqslant \int_{a}^{a+h} \frac{1}{h^p}\Big|\operatorname*{ess\,sup}_{x\in(a,a+2h)} g - \operatorname*{ess\,inf}_{x\in(a,a+2h)} g\Big|^p\,\mathrm{d}x,$$

it follows from Lemma 2 that, for some constant $c = c_p > 0$,

$$\int_{-m}^{m} \big|\tau_h(g)(x)\big|^p\,\mathrm{d}x \leqslant c \sup_{n\in\mathbb{N}} \iint\limits_{\substack{\mathbb{R}\times\mathbb{R}\\ |g(x)-g(y)|>\varepsilon_n}} \frac{\varepsilon_n^p}{|x-y|^{p+1}}\,\mathrm{d}x\,\mathrm{d}y. \tag{14}$$

Since $m \geqslant 2$ is arbitrary, (14) shows that

$$\int_{\mathbb{R}} \big|\tau_h(g)(x)\big|^p\,\mathrm{d}x \leqslant c \sup_{n\in\mathbb{N}} \iint\limits_{\substack{\mathbb{R}\times\mathbb{R}\\ |g(x)-g(y)|>\varepsilon_n}} \frac{\varepsilon_n^p}{|x-y|^{p+1}}\,\mathrm{d}x\,\mathrm{d}y. \tag{15}$$

Since (15) holds for all $0 < h < 1$, it follows that $g \in W^{1,p}(\mathbb{R})$ (see e.g. [2, Chapter 8]). □

In order to establish Theorem 1 in dimension $N \geqslant 2$, we need the following

Lemma 3. *Let g be a measurable function on $\mathbb{R}^N$ and $1 \leqslant p < +\infty$. Then*

$$\int\limits_{\mathbb{R}^{N-1}} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}}{}_{|g(x',x_N)-g(x',y_N)|>2\delta} \frac{1}{|x_N-y_N|^{p+1}}\,\mathrm{d}x_N\,\mathrm{d}y_N\,\mathrm{d}x' \leqslant C_{N,p} \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}{}_{|g(x)-g(y)|>\delta} \frac{1}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y, \quad \forall \delta > 0,$$

where $C_{N,p} > 0$ is a constant depending only on N and p.

Proof of Lemma 3. The method used to prove Lemma 3 is standard (see e.g. [1, Chapter 7]). □

Proof of Theorem 1 completed. Set $\varepsilon_n = 2\delta_n$, for all $n \in \mathbb{N}$. Then it follows from Lemma 3 that

$$\sup_{n\in\mathbb{N}} \underset{|g(x',x_N)-g(x',y_N)|>\varepsilon_n}{\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}}\int_{\mathbb{R}}} \frac{\varepsilon_n^p}{|x_N-y_N|^{p+1}}\,\mathrm{d}x_N\,\mathrm{d}y_N\,\mathrm{d}x' \leqslant C_{N,p} \sup_{n\in\mathbb{N}} \underset{|g(x)-g(y)|>\delta_n}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta_n^p}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y < +\infty.$$

Using Fatou's lemma and Theorem 1 in the case $N=1$, it is not difficult to prove that $g(x',\cdot) \in W^{1,p}(\mathbb{R})$ and moreover

$$\lim_{n\to\infty} \underset{|g(x',x_N)-g(x',y_N)|>\varepsilon_n}{\int_{\mathbb{R}}\int_{\mathbb{R}}} \frac{\varepsilon_n^p}{|x_N-y_N|^{p+1}}\,\mathrm{d}x_N\,\mathrm{d}y_N = C_p \int_{\mathbb{R}} \left|\frac{\partial g}{\partial x_N}(x',x_N)\right|^p \mathrm{d}x_N,$$

for almost everywhere $x' \in \mathbb{R}^{N-1}$ (see [5]).

Thus

$$\int_{\mathbb{R}^N} \left|\frac{\partial g}{\partial x_N}(x)\right|^p \mathrm{d}x = \int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}} \left|\frac{\partial g}{\partial x_N}(x',x_N)\right|^p \mathrm{d}x_N\,\mathrm{d}x' < +\infty.$$

Similarly,

$$\int_{\mathbb{R}^N} \left|\frac{\partial g}{\partial x_i}(x)\right|^p \mathrm{d}x < +\infty, \quad \forall 1 \leqslant i \leqslant N-1.$$

Therefore, $g \in W^{1,p}(\mathbb{R}^N)$ (see e.g. [4, Chapter 4]). □

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