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Mathematical Analysis

Duality of the space of germs of harmonic vector fields on a compact

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Abstract

A vector field $\bar{u}$ in $\mathbb{R}^3$ is said to be harmonic in the open set U if $\operatorname{rot}\bar{u} = \bar{0}$, $\operatorname{div}\bar{u} = 0$ in U. Harmonic vector fields are a natural extension to $\mathbb{R}^3$ of the concept of analytic function of complex variable. We characterize continuous linear functionals acting on the space $h(K)$ of germs of harmonic vector fields on a compact set K. This result provides an $\mathbb{R}^3$-analog of a theorem by G. Köthe on the dual of the space of germs of analytic functions of complex variable on a compact. ***To cite this article: R. Dáger, A. Presa, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Dualité de l'espace des germes des champs de vecteurs harmoniques sur un compact. On dit qu'un champ de vecteurs $\bar{u}$ dans $\mathbb{R}^3$ est harmonique dans un ouvert $U \subset \mathbb{R}^3$ si $\operatorname{rot}\bar{u} = 0$, $\operatorname{div}\bar{u} = 0$ dans U. Les champs de vecteurs harmoniques constituent une extension naturelle dans $\mathbb{R}^3$ des fonctions analitiques complexes. Nous caractérisons les functionels linéares et continues qui agissent dans l'espace $h(K)$ des germes des champs de vecteurs harmoniques sur un compact K. Ce résultat est l'analogue dans $\mathbb{R}^3$ du théorème de G. Köthe sur le dual de l'espace des germes des fonctions analytiques de variable complexe sur un compact. ***Pour citer cet article : R. Dáger, A. Presa, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

1. Introduction

A vector field $\bar{u}$ defined on an open set $U \subset \mathbb{R}^3$ is said to *harmonic* in U if it verifies the equations

$$\operatorname{rot}\bar{u} = 0, \qquad \operatorname{div}\bar{u} = 0 \tag{1}$$

in U. Harmonic vector fields are a natural generalization of analytic functions of complex variable. Indeed, $u(x, y) = u_1(x, y) + \mathrm{i}u_2(x, y)$ is $\mathbb{C}$-analytic in $\mathcal{O} \subset \mathbb{C}$ if and only if the field $\bar{u} = (u_1, -u_2, 0)$ is harmonic in $\mathcal{O}' \times \mathbb{R}$, where $\mathcal{O}' = \{(x, y) \in \mathbb{R}^2\colon x + \mathrm{i}y \in \mathcal{O}\}$.

In recent years several attempts have been made to find the corresponding analogues for harmonic vector fields of important properties of analytic functions of complex variable (see, [2,4,5] and the references therein). However, these

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generalizations are not straightforward, since system (1) is not of surjective symbol, and this implies that the corresponding solutions are not source-type representable, in spite of what happens for other multi-dimensional extensions of analytic functions of complex variable as analytic functions in $\mathbb{C}^N$ or the left-regular quaternionic functions.

It was proved by Koëthe [3] that the dual of the space of germs of analytic functions on a compact set in the complex plane endowed with the inductive limit topology is isomorphic to the to space of analytic functions on the complementary K^c of K that vanish at infinity. Our aim is to investigate whether a similar result holds for harmonic vector fields. To this end we provide in Theorem 3.1 a characterization of continuous linear functionals on the space $h(K)$ of germs of harmonic vector fields on a compact $K \subset \mathbb{R}^3$ in terms of the so-called holomorphic pairs in K^c introduced in [1]. A pair $(f, \bar{v})$ where f is a function and $\bar{v}$ a vector field, defined both in the open set $\mathcal{O}$, is said to be a *holomorphic pair* in $\mathcal{O}$ if

$$\operatorname{rot} \bar{v} = \operatorname{grad} f, \qquad \operatorname{div} \bar{v} = 0.$$

In spite of what happens in the complex case, the dual of the space of germs of harmonic vector fields on a compact cannot be identified with the space $h_0(K^c)$ of harmonic vector fields in the complementary of the compact that vanish at infinity. So, we construct in Theorem 3.2 a space whose dual may be indeed identified with $h_0(K^c)$.

2. The spaces $h(K)$ and $\bar{p}(K)$

For an open set U we denote by $h(U)$, $p(U)$ and $wh(U)$ the locally convex topological vector spaces of all harmonic vector fields in U, holomorphic vector fields in U and harmonic functions (i.e., those θ satisfying $\Delta\theta = 0$) in U, respectively. All these spaces are endowed with the corresponding uniform convergence topologies. Note that for any open U the components of harmonic vector fields and holomorphic pairs in U are harmonic functions, i.e., $h(U) \subset (wh(U))^3$, $p(U) \subset (wh(U))^4$. Further, $\bar{p}(U)$ is the quotient space of $p(U)$ with respect to the closed subspace formed by holomorphic pairs of the form $(0, \operatorname{grad} h)$ with $h \in wh(U)$ with the natural quotient topology. The equivalence class generated by a pair $(f, \bar{v})$ will be denoted by $[f, \bar{v}]$.

Now let K be a compact set and define

$$H(K) = \bigcup_{K \subset U} h(U).$$

In $H(K)$ we introduce the following equivalence relation: $\bar{u}_1, \bar{u}_2 \in H(K)$ are said to be equivalent if they coincide in some open set containing K. Finally, $h(K)$ is the quotient space of $H(K)$ with respect to this equivalence relation. The space $h(K)$ is endowed with the inductive limit topology, that is, the strongest locally convex topology that makes all the canonical maps $r_U : h(U) \to h(K)$ continuous. The elements of $h(K)$ are called the *germs of harmonic vector fields* on K.

Similarly, we define the space $\bar{p}(K)$ of germs of classes of holomorphic pairs on a K and the space $wh(K)$ of germs of harmonic functions on K as inductive limits of $\bar{p}(U)$ and $wh(U)$, respectively. It holds that $h(K) \subset (wh(K))^3$ with continuous imbedding.

Finally, $h_0(U)$ and $p_0(U)$ denote the spaces of those elements of $h(U)$ and $p(U)$ that vanish at infinity.

3. Main results

In the following $\bar{a} \times \bar{b}$ and $\bar{a} \cdot \bar{b}$ denote the vector and scalar products of the vectors $\bar{a}, \bar{b}$, while $\lambda\bar{a}$ is the product of $\lambda \in \mathbb{R}$ by $\bar{a}$.

Theorem 3.1. *Let $K \subset \mathbb{R}^3$ be a compact set and $\Lambda : h(K) \to \mathbb{R}$. Then, $\Lambda \in (h(K))^*$ if and only if, there exists a pair $(f, \bar{v}) \in p_0(K^c)$ such that*

$$\Lambda\big(r_U(\bar{u})\big) = \Lambda_{U,(f,\bar{v})}(\bar{u}) := \frac{1}{4\pi} \int_{\partial K_1} \big(\bar{v} \times (\bar{n} \times \bar{u}) - \bar{n} \cdot (f\bar{u})\big)\, \mathrm{d}S, \tag{2}$$

for every open neighborhood U of K, any vector field $\bar{u} \in h(U)$ and any regular compact K_1 satisfying $K \subset \mathring{K}_1 \subset K_1 \subset U$. Moreover, (a) *if $h \in wh(K^c)$ the functionals $\Lambda_{U,(f,\bar{v})}$ and $\Lambda_{U,(f,\bar{v}+\operatorname{grad} h)}$ coincide on $h(U)$*; (b) *for a given $\Lambda \in (h(K))^*$ all the pairs $(f, \bar{v}) \in p_0(K^c)$ satisfying* (2) *belong to the same equivalence class in $\bar{p}(K^c)$.*

Theorem 3.2. *Let $K \subset \mathbb{R}^3$ be a compact set and $\Lambda : \bar{p}(K) \to \mathbb{R}$. Then, $\Lambda \in (\bar{p}(K))^*$ if and only if, there exists a vector field $\bar{u} \in h_0(K^c)$ such that*

$$\Lambda\big(r_U\big([f, \bar{v}]\big)\big) = \Lambda_{U,\bar{u}}\big([f, \bar{v}]\big) := \frac{1}{4\pi} \int_{\partial K_1} \big(\bar{v} \times (\bar{n} \times \bar{u}) - \bar{n} \cdot (f \bar{u})\big) \, \mathrm{d}S, \tag{3}$$

for every open neighborhood U of K, any pair $(f, \bar{v}) \in p(U)$ and any regular compact K_1 satisfying $K \subset \mathring{K}_1 \subset K_1 \subset U$. Moreover, for a given $\Lambda \in (\bar{p}(K))^*$ there exists a unique $\bar{u} \in h(K^c)$ satisfying (3).

Corollary 3.3. *For every compact $K \subset \mathbb{R}^3$ we have the following isomorphisms*

$$\big(h(K)\big)^* \cong \bar{p}_0\big(K^c\big), \qquad \big(\bar{p}(K)\big)^* \cong h_0\big(K^c\big).$$

Finally, the following consequence of Theorem 3.1 characterizes the fields in $h_0(K^c)$.

Corollary 3.4. $\bar{u} \in h_0(K^c)$ *if and only if there exist a scalar function f and a vector field $\bar{v}$, vanishing both at infinity, such that $\Delta f = 0$, $\Delta \bar{v} = \bar{0}$,* $\operatorname{div} \bar{v} = 0$ *and* $\bar{u} = \operatorname{grad} f + \operatorname{rot} \bar{v}$ *in K^c.*

4. Sketch of the proof of Theorem 3.1

In this section we briefly outline the main ideas used in the proof of Theorem 3.1. The proof of Theorem 3.2 is based on similar ideas.

Proposition 4.1 *(Cauchy–Green representation formulas). Let $U \subset \mathbb{R}^3$ be an open set and L a regular compact subset of U (here and in what follows this means that L has C^∞-boundary) and denote $\phi(x) = \|x\|^{-1}$. Then,*

– *for any vector field $\bar{u} \in h(U)$ and every $x \in \mathring{L}$*

$$\bar{u}(x) = -\frac{1}{4\pi} \int_{\partial L} \big\{\nabla_x \phi(x-y) \times \big(\bar{n}_y \times \bar{u}(y)\big) + \nabla_x \phi(x-y)\big(\bar{n}_y \cdot \bar{u}(y)\big)\big\} \, \mathrm{d}S_y, \tag{4}$$

– *for any pair $(f, \bar{v}) \in p(U)$ and every $x \in \mathring{L}$*

$$\bar{v}(x) = -\frac{1}{4\pi} \int_{\partial L} \big\{\nabla_x \phi(x-y) \times \big(\bar{n}_y \times \bar{v}(y) + f(y)\bar{n}_y\big) + \nabla_x \phi(x-y)\big(\bar{n}_y \cdot \bar{v}(y)\big)\big\} \, \mathrm{d}S_y, \tag{5}$$

$$f(x) = -\frac{1}{4\pi} \int_{\partial L} \nabla_x \phi(x-y) \cdot \big(\bar{n}_y \times \bar{v}(y) + f(y)\bar{n}_y\big) \, \mathrm{d}S_y, \tag{6}$$

where $\bar{n}$ denotes the unit outward normal field to L.

A more general differential form version of (4) is proved in [5] for harmonic differential forms. Formulas (5), (6) are proved using the same ideas.

Now, let L be a regular compact and $\bar{u}$ be a vector field and $(f, \bar{v})$ a pair defined both on ∂L. We denote

$$\Psi(\bar{u}, f, \bar{v}, L) = \frac{1}{4\pi} \int_{\partial L} \big\{\bar{v} \times \big(\bar{n} \times \bar{u}\big) - \bar{n} \cdot (f \bar{u})\big\} \, \mathrm{d}S.$$

We use the following properties of Ψ.

Proposition 4.2. *Let U be an open set containing L and $\bar{u} \in h(U)$. It holds*:

(a) *if $(f, \bar{v}) \in p(U)$ then $\Psi(\bar{u}, f, \bar{v}, L) = 0$,*
(b) *if $h \in wh(U)$ then $\Psi(\bar{u}, f, \bar{v}, L) = \Psi(\bar{u}, f, \bar{v} + \operatorname{grad} h, L)$.*

Let us proceed to the proof of Theorem 3.1. Let $(f,\bar{v}) \in p(K^c)$. For every open U containing K, $\Lambda_{U,(f,\bar{v})}$ is a continuous linear functional on $h(U)$. In view of Proposition 4.2(a), $\Lambda_{U,(f,\bar{v})}$ does not depend on a particular choice of K_1. Besides, if U_1, U_2 are open neighborhoods of K, $\bar{u}_1 \in h(U_1)$ and $\bar{u}_2 \in h(U_2)$ with $r_{U_1}(\bar{u}_1) = r_{U_2}(\bar{u}_2)$ then $\Lambda_{U_1,(f,\bar{v})}(\bar{u}_1) = \Lambda_{U_2,(f,\bar{v})}(\bar{u}_2)$. This implies that the family of continuous linear functionals $\Lambda_{U,(f,\bar{v})} : h(U) \to \mathbb{R}$ defines a continuous linear functional Λ on $h(K)$ satisfying $\Lambda \circ r_U = \Lambda_{U,(f,\bar{v})}$.

On the other hand, using representation formula (4), it may be proved that for every $\Lambda \in (h(K))^*$ and any open U containing K,

$$\Lambda \circ r_U = \Lambda_{U,(f_\Lambda,\bar{v}_\Lambda)},$$

where $(f_\Lambda,\bar{v}_\Lambda) \in p_0(K^c)$ is given by $f_\Lambda(x) = \operatorname{div}_x(\tilde{\Lambda}_y[\phi])$, $\bar{v}_\Lambda(x) = \operatorname{rot}_x(\tilde{\Lambda}_y[\phi])$. Here $\tilde{\Lambda}$ is an extension of Λ to $(wh(K))^3$ and $[\phi]$ is the class in $h(K)$ containing ϕ.

Assertion (a) of Theorem 3.1 follows immediately from Proposition 4.2(b). Assertion (b) is a consequence of the fact that, for any open U containing K, any rectifiable curve γ with support in the complementary $\overline{U}^c$ of the closure of U and any point $c \in \overline{U}^c$ the vector fields

$$\bar{u}_\gamma(x) := \int_\gamma \operatorname{rot}_x \phi(x-y)\cdot \mathrm{d}\bar{y}, \qquad \bar{u}_c(x) := \operatorname{grad}_x \phi(x-c)$$

are harmonic in U and it holds

$$\Lambda_{U,(f,\bar{v})}(\bar{u}_\gamma) = \int_\gamma \bar{v}\cdot \mathrm{d}\bar{x}, \qquad \Lambda_{U,(f,\bar{v})}(\bar{u}_c) = f(c).$$

Thus, if $\Lambda_{U,(f_1,\bar{v}_1)} = \Lambda_{U,(f_2,\bar{v}_2)}$ for every U containing K, then $\overline{w} = \bar{v}_1 - \bar{v}_2$ satisfies $\int_\gamma \overline{w}\cdot \mathrm{d}\bar{x} = 0$ for any rectifiable $\gamma \subset K^c$, so $\overline{w} = \operatorname{grad} h$ for some function h that is harmonic in K^c. Besides, $f = f_1 - f_2 \equiv 0$ in K^c, hence $[f_1,\bar{v}_1] = [f_2,\bar{v}_2]$.

References

[1] B. Gustafsson, D. Khavinson, On annihilators of harmonic vector fields, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 232 (1996); Issled. Linein. Oper. i Teor. Funktsii. 24 (1996) 90–108; English transl. in J. Math. Sci. (N.Y.) 92 (1) (1998) 3600–3612.

[2] A. Herrera Torres, A. Presa, Complex Var. Theory Appl. 50 (2) (2005) 89–101.

[3] G. Köthe, Dualität in der Funktionentheorie, J. Reine Angew Math. 191 (1953) 30–49.

[4] E. Malinnikova, Measures orthogonal to the gradients of harmonic functions, in: Complex Analysis and Dynamical Systems, in: Contemp. Math., vol. 364, Amer. Math. Soc., Providence, RI, 2004, pp. 181–192.

[5] A. Presa Sague, V. Khavin, Uniform approximation by harmonic differential forms on Euclidean space, Algebra i Analiz 7 (6) (1995) 104–152; English transl. in St. Petersburg Math. J. 7 (6) (1996) 943–977.