

Number Theory/Geometry

Three examples of three-dimensional continued fractions in the sense of Klein

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Abstract

The problem of the investigation of the simplest n -dimensional continued fraction in the sense of Klein for $n \geq 2$ was posed by V. Arnold. The answer for the case $n = 2$ can be found in the works of E. Korkina (1995) and G. Lachaud (1995). In present Note we study the case $n = 3$. **To cite this article:** O. Karpenkov, *C. R. Acad. Sci. Paris, Ser. I 343 (2006)*.

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Résumé

Trois exemples des fractions continues trois-dimensionnel en sens de Klein. Le problème de l'étude les plus simple fractions continues n -dimensional en sens de Klein pour $n \geq 2$ a été poser de V. Arnold. Le solution pour la case de $n = 2$ a présenté dans les articles de E. Korkina (1995) et G. Lachaud (1995). Dans la Note présente, on étudie la case de $n = 3$. **Pour citer cet article :** O. Karpenkov, *C. R. Acad. Sci. Paris, Ser. I 343 (2006)*.

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1. Definitions

A point of \mathbb{R}^{n+1} is called *integer* if all its coordinates are integers. A hyperplane is called *integer* if all its integer vectors generate an n -dimensional sublattice of integer lattice. Consider some integer hyperplane and an integer point in the complement to this plane. Let the Euclidean distance from the given point to the given plane equal l . The minimal value of nonzero Euclidean distances from integer points of the space \mathbb{R}^{n+1} to the plane is denoted by l_0 . The ratio l/l_0 is said to be the *integer distance* from the given integer point to the given integer hyperplane.

2. Definition of multidimensional continued fraction in the sense of Klein

Consider arbitrary $n+1$ hyperplanes in \mathbb{R}^{n+1} that intersect at the unique point: at the origin. Assume also that all the given planes do not contain any integer point different to the origin. The complement to these hyperplanes consists

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of 2^{n+1} open orthants. Consider one of these orthants. The boundary of the convex hull of all integer points except the origin in the closure of the orthant is called the *sail* of the orthant. The set of all 2^{n+1} sails is called the *n-dimensional continued fraction* constructed accordingly to the given $n+1$ hyperplanes. Two n -dimensional continued fractions are said to be *equivalent* if there exists a linear lattice preserving transformation of \mathbb{R}^{n+1} taking all sails of one continued fraction to the sails of the other continued fraction.

We associate to any hyperbolic irreducible operator A of $SL(n+1, \mathbb{Z})$ an n -dimensional continued fraction constructed according to the set of all $n+1$ eigen-hyperplanes for A . Any sail of such continued fraction is homeomorphic to \mathbb{R}^n . From Dirichlet unity theorem it follows that the group of all $SL(n+1, \mathbb{Z})$ -operators commuting with A and preserving the sails is homeomorphic to \mathbb{Z}^n and its action is free (we denote this group by $\mathcal{E}(A)$). A *fundamental domain* of the sail with respect to the action of the group $\mathcal{E}(A)$ is a face union that contains exactly one face of the sail from each orbit. (For more information see [1–5].)

3. The examples

Denote by $A_{a,b,c,d}$ the following integer operator

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & d \end{pmatrix}.$$

Example 1. Consider the operator $A_1 = A_{1,-3,0,4}$. The group $\mathcal{E}(A_1)$ is generated by the operators $B_{11} = A_1^{-2}$, $B_{12} = (A_1 - E)^2 A_1^{-2}$, and $B_{13} = (A_1 - E)^2 (A_1 + E) A_1^{-2}$. Let us enumerate all three-dimensional faces for one of the fundamental domains of the sail containing the vertex $(0, 0, 0, 1)$. Let $V_{10} = (-3, -2, -1, 1)$, $V_{1,4i+2j+k} = B_{11}^i B_{12}^j B_{13}^k (V_{10})$ for $i, j, k \in \{0, 1\}$. One of the fundamental domains of the sail contains the following three-dimensional faces: $T_{11} = V_{10} V_{12} V_{14} V_{15}$, $T_{12} = V_{12} V_{14} V_{15} V_{16}$, $T_{13} = V_{12} V_{15} V_{16} V_{17}$, $T_{14} = V_{12} V_{13} V_{15} V_{17}$, $T_{15} = V_{10} V_{12} V_{13} V_{15}$, $T_{16} = V_{10} V_{11} V_{13} V_{15}$, and $T_{17} = V_{10} V_{11} V_{12} V_{13}$ (see Fig. 1 (left)). All listed tetrahedra are taken by some integer affine transformations to the unit basis tetrahedron. The integer distance from the origin to the planes containing the faces T_{11}, \dots, T_{17} equal 4, 3, 2, 4, 3, 2, and 1, respectively.

Statement 1. The continued fraction constructed for any hyperbolic matrix of $SL(4, \mathbb{Z})$ with irreducible characteristic polynomial over rationals and with the sum of absolute values of the elements smaller than 8 is equivalent to the continued fraction of Example 1.

Statement 2. The symmetry (not commuting with A_1) defined by the matrix

$$\begin{pmatrix} 4 & -16 & 17 & -3 \\ 3 & -11 & 11 & -2 \\ 3 & -8 & 6 & -1 \\ 6 & -8 & -2 & 1 \end{pmatrix}$$

acts on the sail of Example 1. This symmetry permutes the equivalence classes (with respect to the action of $\mathcal{E}(A_1)$) of tetrahedra T_{11} and T_{14} , T_{12} and T_{15} , T_{13} and T_{16} , and takes the class of T_{17} to itself.

Example 2. Let us consider the operator $A_2 = A_{1,-4,1,4}$. The group $\mathcal{E}(A_2)$ is generated by the operators $B_{21} = A_2^{-2}$, $B_{22} = (A_2 - E)^2 A_2^{-2}$, and $B_{23} = (A_2 + E) A_2^{-1}$. Let us enumerate all three-dimensional faces for one of the fundamental domains of the sail containing the vertex $(0, 0, 0, 1)$. Let $V_{20} = (-4, -3, -2, 0)$, $V_{2,4i+2j+k} = B_{21}^i B_{22}^j B_{23}^k (V_{20})$ for $i, j, k \in \{0, 1\}$. One of the fundamental domains of the sail contains the following three-dimensional faces: $T_{21} = V_{20} V_{21} V_{23} V_{24}$, $T_{22} = V_{21} V_{23} V_{24} V_{25}$, $T_{23} = V_{20} V_{22} V_{23} V_{24}$, $T_{24} = V_{22} V_{23} V_{24} V_{26}$, $T_{25} = V_{23} V_{24} V_{25} V_{27}$, and $T_{26} = V_{23} V_{24} V_{26} V_{27}$ (see Fig. 1 (middle)). All listed tetrahedra are taken by some integer affine transformations to the unit basis tetrahedron. The integer distance from the origin to the planes containing the faces T_{21}, \dots, T_{26} equal 1, 2, 2, 4, 8, and 13, respectively.

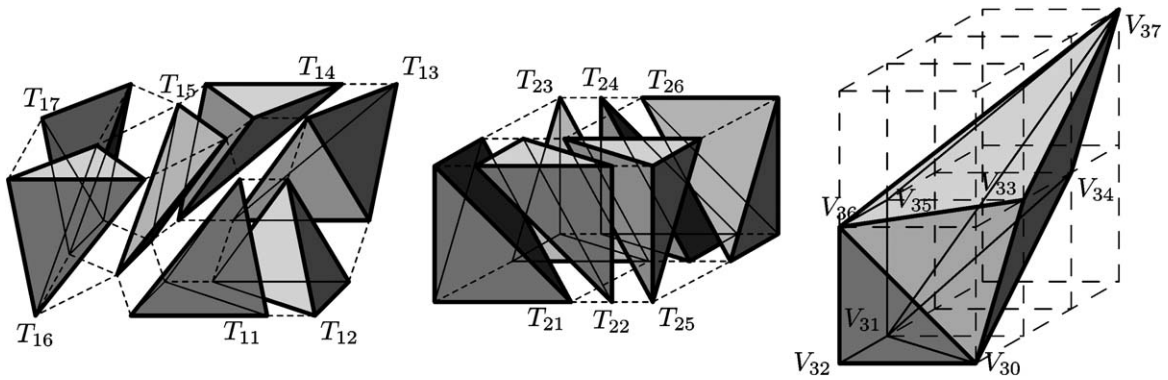


Fig. 1. Gluing faces; see text for details.

Example 3. Consider the operator $A_3 = A_{-1,-3,1,3}$. The group $\mathcal{E}(A_3)$ is generated by the operators $B_{31} = A_3^{-2}$, $B_{32} = (A_3 - E)A_3^{-1}$, and $B_{33} = A_3 + E$. Any fundamental domain of the sail with $(0, 0, 0, 1)$ as a vertex contains a unique three-dimensional face. The polyhedron $V_{30}V_{31}V_{32}V_{33}V_{34}V_{35}V_{36}V_{37}$ shown on Fig. 1 (right) is an example of such face, here $V_{30} = (-1, -1, -1, 0)$, $V_{31} = B_{33}(V_{30})$, $V_{32} = B_{32}B_{33}(V_{30})$, $V_{33} = B_{31}B_{32}^{-1}(V_{30})$, $V_{34} = B_{32}^{-1}(V_{30})$, $V_{35} = B_{31}B_{33}^2(V_{30})$, $V_{36} = B_{31}B_{33}(V_{30})$, $V_{37} = B_{31}B_{32}^{-1}B_{33}(V_{30})$. The described face is contained in the plane on the unit distance from the origin. The integer volume of the face equals 8.

Example 3 provides the negative answer to the following question for the case of $n = 3$: *is it true, that any n -periodic n -dimensional sail contains an n -dimensional face in some hyperplane on integer distance to the origin greater than one?* The answers for $n = 2, 4, 5, 6, \dots$ are unknown. The answer to the following question is also unknown to the author: *is it true, that any n -periodic n -dimensional sail contains an n -dimensional face in some hyperplane on unit integer distance to the origin?*

We show with dotted lines (Fig. 1) how to glue the faces to obtain the combinatorial scheme of the described fundamental domains.

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References

- [1] V.I. Arnold, Continued Fractions, Moscow Center of Continuous Mathematical Education, Moscow, 2002.
- [2] O.N. Karpenkov, On tori decompositions associated with two-dimensional continued fractions of cubic irrationalities, *Funct. Anal. Appl.* 38 (2) (2004) 28–37.
- [3] E.I. Korkina, Two-dimensional continued fractions. The simplest examples, *Proc. Steklov Inst. Math.* 209 (1995) 143–166.
- [4] G. Lachaud, Voiles et Polyèdres de Klein, preprint no. 95-22, Laboratoire de Mathématiques Discrètes du C.N.R.S., Luminy, 1995.
- [5] J.-O. Moussafir, Voiles et Polyèdres de Klein: Geometrie, Algorithmes et Statistiques, docteur en sciences thèse, Université Paris IX-Dauphine, 2000.