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Dynamical Systems

Fractal analysis of spiral trajectories of some vector fields in $\mathbb{R}^3$

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Abstract

We study box dimension and Minkowski content of spiral solutions of some dynamical systems in $\mathbb{R}^3$. ***To cite this article: D. Žubrinić, V. Županović, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Analyse fractale des trajectoires spirales de quelques champs de vecteurs dans $\mathbb{R}^3$. Nous étudions la 'box dimension' et le contenu de Minkowski des solutions spirales de quelques systèmes dynamiques dans $\mathbb{R}^3$. ***Pour citer cet article : D. Žubrinić, V. Županović, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Nous nous sommes interéssés au calcul de la dimensions dite 'box dimension' $\dim_B \Gamma$ des trajectoires spirales de longueur infinie Γ de certain champs de vecteurs dans $\mathbb{R}^3$. On peut considérer cette valeur comme une mesure de concentration dimensionnelle de Γ près de son ensemble limite. Soit Γ une partie de trajectoire du système

$$\begin{aligned} \dot r &= c_1 r^3 + \cdots + c_m r^{2m+1}, \\ \dot\varphi &= 1, \\ \dot z &= d_2 z^2 + \cdots + d_n z^n \end{aligned} \tag{1}$$

près de l'origine dans $\mathbb{R}^3$. Le système (1) est obtenu du système polynomial de coordonnées cartésiens. Soient k et p les entiers positifs minimaux tels que $c_k \neq 0$ et $d_p \neq 0$. Supposons que $c_k d_p > 0$. Si $2k+1 \geqslant p$ alors

$$\dim_B \Gamma = 2 - \frac{2}{2k+1},$$

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tandis que pour $2k + 1 < p$ nous avons

$$\dim_B \Gamma = 2 - \frac{2k + p - 1}{2kp}.$$

Supposons maintenant que (1) ait un cycle limite $r = a$ de multiplicité j, $1 \leqslant j \leqslant m$. Désignons par Γ_1 et Γ_2 les parties des deux trajectoires de (1) près de ce cycle limite vu de l'extérieur et de l'intérieur respectivement. Si $j = 1$ alors

$$\dim_B \Gamma_i = 1, \quad i = 1, 2. \tag{2}$$

Si $j > 1$ alors pour $j \geqslant p$ nous avons

$$\dim_B \Gamma_i = 2 - \frac{1}{j}, \quad i = 1, 2,$$

tandis que pour $j < p$ on a

$$\dim_B \Gamma_i = 2 - \frac{1}{p}, \quad i = 1, 2.$$

D'autre part, nous introduisons une classe de systèmes dans $\mathbb{R}^3$ dont les dimensions des trajectoires dépendent des coefficients du système. Par exemple, pour chaque solution spirale de

$$\dot{r} = a_1 r z, \quad \dot{\varphi} = 1, \quad \dot{z} = b_2 z^2 \tag{3}$$

nous avons

$$\dim_B \Gamma = \frac{2}{1 + a_1/b_2}. \tag{4}$$

Toutes les spirales Γ considerées dans cet article (sauf dans (2), où $\mathcal{M}^1(\Gamma) = \infty$) possèdent le contenu de Minkowski non-dégénéré, c'est-à-dire, il existe $d := \dim_B \Gamma$ et on a $\mathcal{M}_*^d(\Gamma)$, $\mathcal{M}^{*d}(\Gamma) \in]0, \infty[$.

1. Introduction

The aim of this Note is to report about recent progress in finding the box dimension of various spirals appearing as solutions of some dynamical systems in $\mathbb{R}^3$, see Žubrinić, Županović [20]. The box dimension of a spiral can be viewed as a measure of its dimensional concentration near the corresponding limit set. In the case of planar vector fields this question is studied in [19].

As to the history of similar problems, we mention that in Dupain, Mendès France, Tricot [3] the question of Steinhaus dimension of planar spirals is considered. Among the earliest results related to the question of box dimension of planar spirals we cite Tricot [15, p. 122].

We are also interested in Minkowski contents of spirals. Minkowski contents have proved to be important in several instances. We mention the generalized Weyl–Berry conjecture, see He, Lapidus [8], Lapidus, Pomerance [10], and the references therein. In the works of Pašić [13] and Pašić, Županović [12] the question of box dimension of graphs of solutions of the one-dimensional p-Laplace equation is studied using estimates of Minkowski contents. Singular sets of Sobolev functions are studied in Žubrinić [16], and maximally singular Sobolev functions in Horvat, Žubrinić [9], see also [17,18] for related problems.

Here we deal with a class of systems (1) such that the corresponding linear part in Cartesian coordinates has a pure imaginary pair and a simple zero eigenvalues, see Theorems 2.1 and 2.2. Qualitative properties of such systems are treated in Guckenheimer, Holmes [7, Section 7.4]. We also indicate a class of systems in $\mathbb{R}^3$ such that the box dimension of spiral trajectories depends in nontrivial way on the coefficients of the system, see (3) and (4).

All spirals in $\mathbb{R}^3$ are assumed to be contained in a two-dimensional surface. According to properties of the surface, we distinguish Lipschitzian spirals (see Theorem 3.1) and Hölderian spirals (see Theorem 3.2). Furthermore, according to the type of the limit set, a given spiral can be of focus and limit cycle types. We thus obtain four types of spirals in $\mathbb{R}^3$: Lipschitz-focus spirals, Lipschitz-cycle spirals, Hölder-focus spirals and Hölder-cycle spirals. A basic tool in the study of Minkowski content of spirals in the space is provided by Theorem 4.1, which states that nondegeneracy of a set (defined below in this section) is not affected by bi-Lipschitz mappings. It is interesting to note that only one of two projections of a spiral Γ appearing in Theorems 3.1 and 3.2 onto horizontal and vertical planes has box dimension equal to $\dim_B \Gamma$. For example, the projection of spiral Γ in Theorem 3.2(a) onto (y,z)-plane is the graph of the function $y = z^{1/\beta}\sin(z^{-1/\alpha\beta})$. Box dimension of graphs of such functions has been found by Tricot [15, p. 122], which in this case is equal to $2 - \frac{\alpha(1+\beta)}{1+\alpha\beta}$. Surprisingly, this value is equal to $\dim_B \Gamma$.

Let us introduce some notation and terminology. Assume that A is a bounded set in $\mathbb{R}^N$. By $d(x,A)$ we denote Euclidean distance from x to A. The Minkowski sausage of radius ε around A is defined as the ε-neighbourhood of A, that is, as the set $A_\varepsilon := \{y \in \mathbb{R}^N\colon d(y,A) < \varepsilon\}$. We introduce the lower s-dimensional Minkowski content of A, $s \geqslant 0$, by $\mathcal{M}_*^s(A) := \liminf_{\varepsilon\to 0}\frac{|A_\varepsilon|}{\varepsilon^{N-s}}$, where $|\cdot|$ is N-dimensional Lebesgue measure. Similarly we can define the upper s-dimensional Minkowski content of A. The corresponding lower and upper box dimensions of A are denoted by $\underline{\dim}_B A$ and $\overline{\dim}_B A$. See Falconer [5] or Mattila [11] for various properties of box dimensions. If A is such that $\underline{\dim}_B A = \overline{\dim}_B A$, the common value is denoted by $d := \dim_B A$. Furthermore, if both the upper and lower d-dimensional Minkowski contents of A are contained in $]0,\infty[$, we say that the set A is nondegenerate, or that it has nondegenerate Minkowski content.

2. Box dimension of spiral trajectories of some vector fields in $\mathbb{R}^3$

We consider dynamical system (1) defined in cylindrical coordinates, where $m, n \in \mathbb{N}$ and $c_i, d_i \in \mathbb{R}$. The first two equations represent a standard planar Hopf–Takens bifurcation model obtained from polynomial system in Cartesian coordinates, see Takens [14]. See also Caubergh, Dumortier [1], Caubergh, Françoise [2] for some generalizations of Takens' results.

Theorem 2.1 *(Spiral solutions of focus type). Let Γ be a part of a trajectory of* (1) *near the origin. Assume that k and p are minimal positive integers such that $c_k \neq 0$ and $d_p \neq 0$. Assume also that $c_k d_p > 0$.*

(a) *If $2k+1 \geqslant p$ then Γ is a Lipschitz-focus and nondegenerate spiral with* $\dim_B \Gamma = 2 - \frac{2}{2k+1}$.
(b) *If $2k+1 < p$ then Γ is a Hölder-focus and nondegenerate spiral with* $\dim_B \Gamma = 2 - \frac{2k+p-1}{2kp}$.

Theorem 2.2 *(Spiral solutions of limit cycle type). Let the system* (1) *have limit cycle $r = a$ of multiplicity j, $1 \leqslant j \leqslant m$. By Γ_1 and Γ_2 we denote the parts of two trajectories of* (1) *near the limit cycle from outside and inside respectively. For $j = 1$ we have* $\dim_B \Gamma_i = 1$, $i = 1, 2$. *For $j > 1$ we have the following*:

(a) *if $j \geqslant p$ then Γ_i are Lipschitz-cycle nondegenerate spirals, and* $\dim_B \Gamma_i = 2 - \frac{1}{j}$, $i = 1, 2$;
(b) *if $j < p$ then Γ_i are Hölder-cycle nondegenerate spirals, and* $\dim_B \Gamma_i = 2 - \frac{1}{p}$, $i = 1, 2$.

It is worth noting that box dimensions of spirals in Theorems 2.1 and 2.2 depend only on the exponents appearing on right-hand sides of (1). On the other hand, in (3) we introduce an example of a class of dynamical systems such that the box dimension of its spiral trajectories depends in nontrivial way on the coefficients of the system.

Example 1. Let us consider the system (3). Its solution is

$$r = C_1(-b_2t + C_3)^{-a_1/b_2}, \quad \varphi = t + C_2, \quad z = \frac{1}{-b_2t + C_3}.$$

Note that the corresponding spiral Γ near the origin is contained on the surface $z = C \cdot r^{b_2/a_1}$. If $a_1/b_2 \in]0,1[$ then Γ is nondegenerate and $\dim_B \Gamma = \frac{2}{1+a_1/b_2}$. Indeed, the claim follows from Theorem 3.1(a) below, since $\alpha = a_1/b_2 \in]0,1[$, $\beta = b_2/a_1 > 1$. For other cases and a more general result see [20, Example and Proposition 3].

3. Lipschitzian and Hölderian spirals in $\mathbb{R}^3$

The proof of Theorems 2.1 and 2.2 rests on the application of extensions of Theorems 3.1 and 3.2 below, see [20, Theorems 2 and 3] for the general form. The idea is to represent the spiral Γ of Theorem 2.1 in the form $r = f(\varphi)$, $\varphi \geqslant \varphi_1$, $z = g(r)$, where $f(\varphi)$ behaves like $\varphi^{-\alpha}$ with $\alpha := 1/(2k)$ for large φ, and $g(r)$ behaves like r^β with $\beta := 2k/(p-1)$ for small r. In proving Theorem 2.2 we have to change $g(r)$ to $g(|a-r|)$.

Theorem 3.1 *(Lipschitzian spirals).* (a) *Let Γ be a Lipschitz-focus spiral in $\mathbb{R}^3$ defined in cylindrical coordinates by*

$$r=\varphi^{-\alpha},\quad \varphi\geqslant\varphi_1,\quad z=g(r\cos\varphi,r\sin\varphi),\tag{5}$$

where $g:\mathbb{R}^2\to\mathbb{R}$ is any given Lipschitz function defined in Cartesian coordinates. If $\alpha\in\,]0,1[$ then Γ is nondegenerate and $\dim_B\Gamma=\frac{2}{1+\alpha}$.

(b) *Let Γ be a Lipschitz-cycle spiral defined by $r=1$, $\varphi\geqslant\varphi_1$, $z=\varphi^{-\alpha}$. Then for any $\alpha>0$ we have that Γ is nondegenerate and* $\dim_B\Gamma=\frac{2+\alpha}{1+\alpha}$.

Theorem 3.2 *(Hölderian spirals). Assume that $\beta\in\,]0,1[$.* (a) *Let Γ be a Hölder-focus spiral defined by $r=\varphi^{-\alpha}$, $\varphi\in[\varphi_1,\infty)$, $z=r^\beta$, and $\alpha\in]0,1[$. Then Γ is nondegenerate and* $\dim_B\Gamma=\frac{2-\alpha(1-\beta)}{1+\alpha\beta}$.

(b) *Let Γ be a Hölder-cycle spiral defined by $r=1-\varphi^{-\alpha}$, $\varphi\in[\varphi_1,\infty)$, $z=|1-r|^\beta$, $\alpha>0$. Then Γ is nondegenerate and* $\dim_B\Gamma=\frac{2+\alpha\beta}{1+\alpha\beta}$.

Theorems 3.1 and 3.2(b) can be obtained from Theorem 4.1. In the proof of Theorem 3.2(a) we exploit Weyl's tube formula, see Gray [6, p. 6]. In the proof of Theorem 3.1(b) we first consider the auxiliary spiral Γ_0 in (x,y)-plane defined by $r=1-\varphi^{-\alpha}$. It is a subset of the solid torus Ω obtained as ε-neighbourhood of the circle $r=1$, $z=0$, with $\varepsilon=1/2$. The spiral Γ in the theorem can obviously be obtained by "rotation" $F:\Omega\to\Omega$ of the torus around its central circle for $\pi/2$, which is a bi-Lipschitzian operation. More precisely, for a point $P(r_0,\varphi_0,z_0)\in\Omega$ we define $F(P)$ as a point obtained by rotation of P in the plane $\varphi=\varphi_0$ around the point $(1,\varphi,0)$ for $\pi/2$ in positive direction with respect to standard orientation of the circle. The claim of Theorem 3.1(b) then follows from Theorem 4.1. Theorem 3.2(b) can be proved using Theorem 3.1(b) combined with Theorem 4.1. See [20, Theorems 2 and 3] for details.

4. Minkowski content and bi-Lipschitz mappings

The following result represents a refinement of the well known fact that box dimension of a set is left unchanged under bi-Lipschitzian mappings, see Falconer [5, p. 44]. The Jacobian of a Lipschitz mapping F is $J_F(x):=\det F'(x)$. Its L^∞-norm will be denoted by $\|J_F\|_\infty$. In the proof of Theorem 4.1 we exploit a generalized change of variables formula involving Lipschitz mappings, see [20, Theorem 1] and Evans, Gariepy [4, p. 108].

Theorem 4.1. *Let Ω and Ω' be open sets in $\mathbb{R}^N$, and let $F:\Omega\to\Omega'$ be a bi-Lipschitz mapping with lower and upper Lipschitz constants equal to $\underline{C}$ and $\overline{C}$ respectively. Let A be a bounded set such that $\bar{A}\subseteq\Omega$. Then for all $s\geqslant0$ we have*

$$\frac{1}{\overline{C}^{N-s}\|J_{F^{-1}}\|_\infty}\mathcal{M}^{*s}(A)\leqslant\mathcal{M}^{*s}\big(F(A)\big)\leqslant\frac{\|J_F\|_\infty}{\underline{C}^{N-s}}\mathcal{M}^{*s}(A),\tag{6}$$

and analogously for the lower s-dimensional Minkowski content.

Remark 1. It has been shown in [17, Theorem 2] that for any bounded set A in $\mathbb{R}^N$ which is nondegenerate we have that for any $\varepsilon>0$,

$$\int_{A_\varepsilon} d(x,A)^{-\gamma}\,\mathrm{d}x<\infty\quad\Longleftrightarrow\quad\gamma<N-\dim_B A.\tag{7}$$

For generalizations of this result involving generalized Minkowski contents see [18]. The condition of nondegeneracy of A for (7) to hold is indeed essential, see [18, Theorem 4.2]. In Sections 3 and 4 we have obtained many spirals $A=\Gamma$ in $\mathbb{R}^3$ having the desired nondegeneracy property. In this way we obtain new, nontrivial Lebesgue integrable functions having singular sets on spirals. Using such functions we can construct Sobolev functions with singular sets being spirals, see [16].

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