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Partial Differential Equations

Nonlinear Schrödinger equations with potentials vanishing at infinity

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Abstract

In this Note, we deal with stationary nonlinear Schrödinger equations of the form

$$-\varepsilon^2 \Delta u + V(x)u = K(x)u^p, \quad x \in \mathbb{R}^N,$$

where $V, K > 0$ and $p > 1$ is subcritical. We allow the potential V to vanish at infinity and the competing function K to be unbounded. In this framework, positive ground states may not exist. We prove the existence of at least one positive bound state solution in the *semi-classical limit*, i.e. for $\varepsilon \sim 0$. We also investigate the qualitative properties of the solution as $\varepsilon \to 0$. ***To cite this article: D. Bonheure, J. Van Schaftingen, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Équations de Schrödinger non linéaires avec potentiels s'annulant à l'infini. Dans cette Note, nous considérons des équations de Schrödinger non linéaires stationnaires du type

$$-\varepsilon^2 \Delta u + V(x)u = K(x)u^p, \quad x \in \mathbb{R}^N,$$

où $V, K > 0$ et $p > 1$ est sous-critique. Nous considérons un potentiel V qui s'annule éventuellement à l'infini et une fonction de compétition K qui pourrait ne pas être bornée. Dans ce cas, l'existence d'une solution positive d'énergie minimale n'est pas assurée. Nous démontrons l'existence d'au moins une solution positive dans la limite semi-classique, c'est-à-dire pour $\varepsilon \sim 0$. Nous étudions également les propriétés qualitatives de cette solution lorsque $\varepsilon \to 0$. ***Pour citer cet article : D. Bonheure, J. Van Schaftingen, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Nous considérons le problème elliptique non linéaire (2), et nous cherchons des états semi-classiques, c'est-à-dire des solutions pour ε proche de 0. Ce problème a été traité par une réduction de Lyapunov–Schmidt [1,3,11], et

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par des méthodes variationnelles locales [8–10]. Ces solutions se concentrent autour de points critiques de $\mathcal{A}$ défini par (3). Ces travaux supposent que le potentiel V a une borne inférieure strictement positive et que K est borné supérieurement. Ambrosetti, Felli et Malchiodi [2] ont pu traiter le cas où la vitesse de convergence de V vers 0 à l'infini contrôle celle de K vers 0 à l'infini.

Dans cette Note nous traitons le cas où V et K satisfont $(\mathcal{G}^1_\infty)$, $(\mathcal{G}^2_\infty)$ ou $(\mathcal{G}^3_\infty)$. Par ces conditions, la croissance de K à l'infini est également contrôlée par celle de V, avec un facteur supplémentaire qui donne plus de liberté. Par exemple, le potentiel V peut tendre vers 0 à l'infini alors que K tend vers l'infini. Sous ces hypothèses, nous obtenons l'existence de solutions qui se concentrent autour de minima locaux de $\mathcal{A}$.

Théorème 0.1. *Soit $p \in]1, (N+2)/(N-2)[$ si $N \geqslant 3$ ou $p \in]1, \infty[$ sinon. Soient $V, K \in C(\mathbb{R}^N, \mathbb{R}^+)$ tels que l'une des conditions $(\mathcal{G}^i_\infty)$ soit satisfaite. Soit $\Lambda \subset \mathbb{R}^N$ un ouvert borné. Si*

$$\mathcal{A}_0 := \inf_{x \in \Lambda} \mathcal{A}(x) < \inf_{x \in \partial\Lambda} \mathcal{A}(x),$$

alors il existe $\varepsilon_0 > 0$ tel que pour tout $0 < \varepsilon < \varepsilon_0$, l'équation (2) ait au moins une solution strictement positive u_ε. La fonction u_ε atteint son maximum en $x_\varepsilon \in \Lambda$, et $\lim_{\varepsilon \to 0} \mathcal{A}(x_\varepsilon) = \mathcal{A}_0$. De plus, si $N > 4$ ou si (4) a lieu, $\|u_\varepsilon\|_2^2 = \mathrm{O}(\varepsilon^N)$.

Notre preuve se base sur la méthode de pénalisation développée par del Pino et Felmer [8]. Les solutions sont formellement des points critiques de $\mathcal{I}_\varepsilon$ (voir (5)). Afin d'obtenir une fonctionnelle bien définie et la géométrie du col, nous étudions les points critiques de la fonctionnelle pénalisée $\mathcal{J}_\varepsilon$ définie par (6), (7) et (8).

La fonctionnelle $\mathcal{J}_\varepsilon$ a la géométrie du col et satisfait la condition de Palais–Smale, elle a donc un point critique non trivial u_ε. Pour prouver que les suites de Palais–Smale ne perdent pas de masse à l'infini, les estimations sur la norme L^2 de ces suites sont remplacées par l'inégalité de Hardy.

Pour conclure, il reste à montrer que les points critique de $\mathcal{J}_\varepsilon$ ainsi obtenus sont des points critiques de $\mathcal{I}_\varepsilon$. Pour cela il suffit que l'inégalité (9) soit satisfaite pour ε petit. Comme dans [8], u_ε tend uniformément vers 0 sur $\partial\Lambda$. En utilisant des fonctions de comparaison w_i adéquates et le principe du maximum, on montre que l'inégalité (9) est satisfaite pour ε petit.

En ce qui concerne la concentration, les hypothèses de régularité réduites sur V et K permettent de montrer que le maximum global de u_ε est atteint à l'intérieur de Λ et que tous les maxima locaux de u_ε sont dans une boule de rayon $\mathrm{O}(\varepsilon)$ autour de ce point. Par ailleurs, on a des estimations de la forme (10).

Ces résultats sont des cas particuliers d'un résultat plus général [6], où la non-linéarité u^p est remplacée par $f(u)$, avec f continue, $f(s)/s$ croissante et f satisfaisant une condition d'Ambrosetti–Rabinowitz. Nous y considérons aussi l'équation (2) sur un domaine à frontière régulière bornée avec une condition de Dirichlet sur le bord, à condition que V et K satisfassent la condition (11).

1. Introduction

The nonlinear Schrödinger equation

$$\mathrm{i}\hbar \frac{\partial \psi}{\partial t} = -\hbar^2 \Delta\psi + W(x)\psi - K(x)|\psi|^{p-1}\psi = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1}$$

where $\hbar$ denotes the Plank constant and i is the imaginary unit, models the non-relativistic quantic evolution in nonlinear optics or plasma physics. It is expected that classical mechanics can be recovered by letting $\hbar \to 0$ in (1). The limiting behaviour as $\hbar \to 0$ is called *semi-classical* and the study of the dynamics of (1) for small $\hbar$ leads naturally to that of *semi-classical states*, i.e. *standing wave* solutions of the form $\psi(t, x) = \mathrm{e}^{-\mathrm{i}Et/\hbar} u(x)$, where E is the energy of the wave. Mathematically, this amounts to consider the elliptic equation

$$-\varepsilon^2 \Delta u + V(x)u = K(x)u^p, \quad x \in \mathbb{R}^N, \tag{2}$$

where $\varepsilon = \hbar$ and $V(x) = W(x) - E$.

In their pioneering work [11], Floer and Weinstein constructed positive solutions of (2) in the case $p = 3$, $N = 1$ and $K \equiv 1$, assuming that V is a globally bounded potential having a nondegenerate critical point, say $x = 0$, and $\inf V > 0$. They also proved that the solution concentrates around the critical point of V, that is most of the density $|u_\varepsilon|^2$ of the solution is contained in a ball around 0 whose radius is of the order of ε. This result has been extended to higher dimensions under various assumptions using different methods. The Lyapunov–Schmidt reduction scheme proposed by Floer and Weinstein has been further extended and combined with variational arguments by Ambrosetti et al., see e.g. [1,3], while a local variational analysis has been proposed by del Pino and Felmer [8–10]. When K is not constant and $\varepsilon \to 0$, positive solutions of (2) concentrate around critical points of the auxiliary function $\mathcal{A}: \mathbb{R}^N \to \mathbb{R}^+$ defined by

$$\mathcal{A}(x) := \frac{V(x)^{\frac{p+1}{p-1} - \frac{N}{2}}}{K(x)^{\frac{2}{p-1}}}, \tag{3}$$

and a sufficient condition for the existence of a positive least energy solution can be formulated according to the behaviour of this function $\mathcal{A}$, see e.g. [12].

These works all assume that the potential V is bounded away from zero. Recently, Ambrosetti, Felli and Malchiodi [2] bypassed this restriction. However, they assumed either that the potential V is bounded away from 0 at infinity or that the competing function K decays to 0 at infinity with a rate related to that of V. Byeon and Wang considered the case in which V vanishes somewhere but is bounded away from 0 at infinity [7].

In this Note, we consider the case where $V, K \in C(\mathbb{R}^N, \mathbb{R}^+)$ satisfy one of the three following growth conditions at infinity:

- there exist $\alpha < 2$ and $\lambda > 0$ such that

$$(\mathcal{G}^1_\infty) \quad \liminf_{|x|\to\infty} V(x)|x|^\alpha > 0 \quad \text{and} \quad \lim_{|x|\to\infty} \exp\bigl(-\lambda|x|^{1-\alpha/2}\bigr)\frac{K(x)}{V(x)} = 0;$$

- there exists $\lambda > 0$ such that

$$(\mathcal{G}^2_\infty) \quad \liminf_{|x|\to\infty} V(x)|x|^2 > 0 \quad \text{and} \quad \lim_{|x|\to\infty} |x|^{-\lambda}\frac{K(x)}{V(x)} = 0;$$

- $N > 2$ and

$$(\mathcal{G}^3_\infty) \quad \lim_{|x|\to\infty} |x|^{-(p-1)(N-2)}\frac{K(x)}{V(x)} < +\infty.$$

We prove the existence of bound state solutions concentrating around local minima of $\mathcal{A}$:

Theorem 1.1. *Let $p \in \,]1, (N+2)/(N-2)[$ if $N \geqslant 3$ or $p \in \,]1, \infty[$ otherwise. Let $V, K \in C(\mathbb{R}^N, \mathbb{R}^+)$ satisfy one set $(\mathcal{G}^i_\infty)$ of growth conditions. Let $\Lambda \subset \mathbb{R}^N$ be open and bounded. If*

$$\mathcal{A}_0 := \inf_{x\in\Lambda} \mathcal{A}(x) < \inf_{x\in\partial\Lambda} \mathcal{A}(x),$$

then there exist $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, Eq. (2) has at least one positive solution u_ε. The function u_ε attains its maximum at $x_\varepsilon \in \Lambda$, and $\lim_{\varepsilon\to 0} \mathcal{A}(x_\varepsilon) = \mathcal{A}_0$. Moreover if either

$$\liminf_{|x|\to\infty} V(x)|x|^2 > 0 \tag{4}$$

or $N > 4$, then $u_\varepsilon \in L^2(\mathbb{R}^N)$ and $\|u_\varepsilon\|_2^2 = \mathrm{O}(\varepsilon^N)$.

The solutions also satisfy $\int_{\mathbb{R}^N} \varepsilon^2|\nabla u_\varepsilon|^2 + V(x)|u_\varepsilon|^2 = \mathrm{O}(\varepsilon^N)$. Since V does not have a positive lower bound, this does not imply that $\int_{\mathbb{R}^N} |u_\varepsilon|^2$ satisfies the same estimate, or that it is finite.

When both V and K are Hölder-continuous, bounded and bounded away from zero, Theorem 1.1 is essentially due to del Pino and Felmer [8]. If V and K are smoother and if the minimum of $\mathcal{A}$ in Λ is topologically stable, the result has also been proved via perturbation arguments, see e.g. [5].

In the sequel of this Note we shall outline the proof, which is obtained by a penalisation method adapted from that of del Pino and Felmer [8]. We highlight in Section 2 the main changes in that method and the new difficulties. Theorem 1.1 is in fact a special case of a more general existence result [6]. Comments on these generalisations are given in the final section.

While completing this Note, we heard from David Ruiz about a recent preprint by Ambrosetti, Malchiodi and Ruiz [4] in which they consider Eq. (2) under similar assumptions, and they obtain solutions around any stable critical point of the concentration function $\mathcal{A}$. They require K to be bounded from above, and their method also relies on the homogeneity of the nonlinear term $K(x)u^p$.

2. Sketch of the proof of Theorem 1.1

2.1. The penalized functional

Formally, the elliptic equation (2) is the Euler–Lagrange equation of the functional

$$\mathcal{I}_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^N} \left(\varepsilon^2|\nabla u|^2 + V(x)u^2\right) - \frac{1}{p+1}\int_{\mathbb{R}^N} K(x)u^{p+1}. \tag{5}$$

The first integral defines the square of the norm on the Hilbert space $\mathcal{H}_\varepsilon$.

The assumptions on K are to weak to conclude that the second term of $\mathcal{I}_\varepsilon$ is well defined on $\mathcal{H}_\varepsilon$, or that $\mathcal{I}_\varepsilon$ has the mountain-pass geometry. This is circumvented by using a penalization functional introduced by del Pino and Felmer in [8], which consists in modifying the superquadratic term in $\mathcal{I}_\varepsilon$ outside Λ. Let $k > 1$ be a fixed constant and define $\tilde{f} : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ by

$$\tilde{f}(x,u) := u \min\left(\frac{V(x)}{k}, K(x)u^{p-1}\right). \tag{6}$$

Denoting by χ_Λ the characteristic function of the set Λ, we define $g : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ by

$$g(x,u) := \chi_\Lambda(x)K(x)u^p + \left(1 - \chi_\Lambda(x)\right)\tilde{f}(x,u). \tag{7}$$

We introduce a convenient penalized functional

$$\mathcal{J}_\varepsilon(u) := \frac{1}{2}\left(\int_{\mathbb{R}^N} \varepsilon^2|\nabla u|^2 + V(x)u^2\right) - \int_{\mathbb{R}^N} G(x,u), \tag{8}$$

where $G(x,u) := \int_0^u g(x,s)\,\mathrm{d}s$. Under our assumptions, this functional is well-defined in the space $\mathcal{H}_\varepsilon$. From the way our penalized functional is defined, it might seem at first sight that the method of del Pino and Felmer extends to our framework with only minor changes. Whereas this is true for some arguments, the possibility of V to vanish at infinity brings a lot of troubles and requires careful estimates.

2.2. The Palais–Smale condition

By the same arguments as in [8], $\mathcal{J}_\varepsilon$ has a mountain-pass geometry in $\mathcal{H}_\varepsilon$. Therefore, $\mathcal{J}_\varepsilon$ will have a nontrivial critical point if the classical Palais–Smale condition holds. Our assumptions on V do not imply the L^2-boundedness of Palais–Smale sequences, but Hardy's inequality can still prevent losses of mass at infinity in those sequences. Hardy's inequality does not hold in two dimensions, but under the condition $(\mathcal{G}^2_\infty)$ however, $\int_{\mathbb{R}^2} |u|^2/|x|^2$ is controlled by the norm in $\mathcal{H}_\varepsilon$. This is one of the reasons for which we assume $N > 2$ in condition $(\mathcal{G}^3_\infty)$.

2.3. Comparison functions

The mountain-pass critical point u_ε of $\mathcal{J}_\varepsilon$ is a critical point of $\mathcal{I}_\varepsilon$ provided

$$u_\varepsilon^{p-1}(x)K(x)/V(x) \leqslant 1/k \tag{9}$$

for every $x \in \mathbb{R}^N \setminus \Lambda$. As in [8], u_ε tends uniformly to 0 as $\varepsilon \to 0$ on $\partial\Lambda$. By the maximum principle, u_ε tends to 0 uniformly outside Λ. Since K/V is not bounded, this does not imply (9) as in [8]. However, by a suitable maximum principle, critical points of $\mathcal{J}_\varepsilon$ can be compared with supersolutions of the linear operator L_ε defined formally by $L_\varepsilon u = -\varepsilon^2\Delta u + V(x)u$.

We then find *comparison functions* which can be compared with the solution u_ε. Assume, in order to fix ideas, that ($\mathcal{G}^1_\infty$) holds. When ε is small enough, the function $w_1(x) = \exp(-\lambda|x|^{1-\alpha/2})$ is a supersolution of L_ε. Combining this fact with the uniform convergence of u_ε on $\partial\Lambda$, the above mentioned maximum principle and the second part of assumption ($\mathcal{G}^1_\infty$), one concludes that u_ε satisfies (9). When assuming either ($\mathcal{G}^2_\infty$) or ($\mathcal{G}^3_\infty$), we argue in a similar way, using for instance respectively $w_2(x) = |x|^{-\lambda}$ and $w_3(x) = |x|^{2-N}$ as supersolutions in the comparison argument.

2.4. Concentration

In [8], when V is Hölder continuous, it is established that u_ε has a unique local (and hence global) maximum. As we only assume that V and K are continuous, the weakness of the regularity of the solution ruled out the arguments used therein. However, the global maximum $x_\varepsilon \in \Lambda$ is essentially unique in the sense that if y_ε are local maxima of u_ε, then $d(x_\varepsilon, y_\varepsilon)/\varepsilon$ is bounded as $\varepsilon \to 0$.

Using the lower bound on V, one obtains in [8] the decay $u_\varepsilon(x) \leqslant C\exp(-\lambda|x - x_\varepsilon|/\varepsilon)$. When V is not bounded away from zero, we do not recover exponential decay. In some sense, the solution inherits its decay properties from the behaviour of V. According to one of the conditions ($\mathcal{G}^i_\infty$), we have

$$u_\varepsilon(x) \leqslant C w_i\left(\frac{x - x_\varepsilon}{\varepsilon}\right), \tag{10}$$

where w_i is the corresponding comparison function already introduced above and x_ε is a global maximum of u_ε. Such estimates are delicate and depend on comparison arguments uniform in ε. They are obtained by using *families of barrier functions*, i.e. families of comparison functions. The L^2-integrability of u_ε can be deduced from the concentration estimate (10).

3. A more general framework

As pointed out in the introduction, we are able to deal with more general assumptions than those of Theorem 1.1. In fact, u^p in (2) can be replaced by a more general superlinear function $f \in C(\mathbb{R}, \mathbb{R})$ satisfying the usual nonquadraticity condition of Ambrosetti–Rabinowitz and the right monotonicity condition that allows to obtain a suitable variational characterisation of minimisers on the Nehari manifold. We emphasise that we neither require smoothness of f nor uniqueness of ground states of the limiting problems

$$-\Delta u + au = bf(u), \quad a, b \in \mathbb{R}.$$

When u^p is replaced by a general nonlinear term, it is not usually possible to give an explicit expression of the concentration function, i.e. the energy associated the ground state solutions of the limiting equation which is given by $\mathcal{A}$ in the homogeneous case. Also, the growth conditions ($\mathcal{G}^i_\infty$) have to be adapted according to the behaviour of $f(u)/u$ close to zero. We refer to [6] for further detail.

Our method and results can also be extended to open sets with a smooth compact boundary [6]. The potential V and the competing function K should satisfy the growth condition on the boundary

$$\lim_{x \to \partial\Omega} d(x, \partial\Omega)^{p-1} \frac{K(x)}{V(x)} < +\infty. \tag{11}$$

References

[1] A. Ambrosetti, M. Badiale, S. Cingolani, Semiclassical states of nonlinear Schrödinger equations, Arch. Rational Mech. Anal. 140 (1997) 285–300.
[2] A. Ambrosetti, V. Felli, A. Malchiodi, Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity, J. Eur. Math. Soc. 7 (2005) 117–144.
[3] A. Ambrosetti, A. Malchiodi, W.-M. Ni, Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres, Part I, Comm. Math. Phys. 235 (2003) 427–466.
[4] A. Ambrosetti, A. Malchiodi, D. Ruiz, Bound states of nonlinear Schrödinger equations with potentials vanishing at infinity, J. Anal. Math., submitted for publication.
[5] A. Ambrosetti, A. Malchiodi, S. Secchi, Multiplicity results for some nonlinear singularly perturbed elliptic problems on $\mathbb{R}^N$, Arch. Rational Mech. Anal. 159 (2001) 253–271.
[6] D. Bonheure, J. Van Schaftingen, Bound state solutions for a class of nonlinear Schrödinger equations, preprint.

[7] J. Byeon, Z.Q. Wang, Standing waves with a critical frequency for nonlinear Schrödinger equations, Arch. Rational Mech. Anal. 165 (4) (2002) 295–316.
[8] M. del Pino, P. Felmer, Local mountain passes for semilinear elliptic problems in unbounded domains, Calc. Var. 4 (1996) 121–137.
[9] M. del Pino, P. Felmer, Semi classical states for nonlinear Schrödinger equations, J. Funct. Anal. 149 (1) (1997) 245–265.
[10] M. del Pino, P. Felmer, Semi-classical states of nonlinear Schrödinger equations: a variational reduction method, Math. Ann. 324 (1) (2002) 1–32.
[11] A. Floer, A. Weinstein, Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential, J. Funct. Anal. 69 (1986) 397–408.
[12] X. Wang, B. Zeng, On concentration of positive bound states of nonlinear Schrödinger equations with competing potential functions, SIAM J. Math. Anal. 28 (3) (1997) 633–655.