Available online at www.sciencedirect.com
SCIENCE DIRECT®

C. R. Acad. Sci. Paris, Ser. I 342 (2006) 865–868

COMPTES RENDUS MATHEMATIQUE

http://france.elsevier.com/direct/CRASS1/

Differential Geometry

Vector fields dynamics as geodesic motion on Lie groups

Gabriel Teodor Pripoae

University of Bucharest, Faculty of Mathematics and Informatics, 14 Academiei St., Bucharest, Romania

Received 19 December 2005; accepted after revision 1 April 2006

Available online 3 May 2006

Presented by Étienne Ghys

Abstract

We study to what extent vector fields on Lie groups may be considered as geodesic fields. For a given left invariant vector field on a Lie group, we prove there exists a Riemannian metric whose geodesics are its trajectories. When we consider left invariant metrics, differences between the Riemannian and the Lorentzian cases appear, coded by properties of the Lie algebra. ***To cite this article: G.T. Pripoae, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Dynamique des champs vectoriels comme mouvement géodésique des groupes de Lie. On étudie les conditions pour que les champs de vecteurs sur les groupes de Lie devienent des champs géodésiques. Pour un champ de vecteurs invariant à gauche, donné sur un groupe de Lie, on prouve qu'il existe une métrique riemannienne dont les géodésiques en sont les trajectoires. Dans le cas des métriques invariantes, on met en évidence certaines differences entre le cas riemannien et celui lorentzien, codées par des propriétés de l'algèbre de Lie. ***Pour citer cet article : G.T. Pripoae, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction

The modelization of a dynamic system is most clearly expressed as a vector field on a differentiable manifold. The study of its properties, invariants and possible classification criteria may be done with topological, differential or (geo)metrical technics. An old open problem, attributed to H. Poincaré, asks: *Given a vector field X on a differentiable manifold M, does there exist a Riemannian metric g on M such that the trajectories of X be geodesics of g?*

Even if it is expressed in a metric terminology, the true essence of the problem depends only on the differentiable structure of the manifold M. An eventual positive answer for this problem might offer insights for importing Riemannian technics and results, in a canonical way, into the study of vector fields.

Locally, the problem has (obviously) positive answer; globally, it is still open.

The aim of this Note is to solve the problem on Lie groups. The interest in the dynamics of vector fields on Lie groups grew after 1966, when a seminal paper of V. Arnold appeared [1], describing the dynamics of the Euler equation in terms of geodesics on SO(3). For an up-dated review of the topic, see [3]. Our main results state:

E-mail address: gpripoae@fmi.unibuc.ro (G.T. Pripoae).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2006.04.009

Theorem 1. *For every left invariant vector field ξ on a Lie group G, there exists a Riemannian metric g such that the trajectories of ξ be geodesics of g.*

Theorem 2. *Let ξ be a non-vanishing left invariant vector field on a Lie group G. Then there exists a left invariant Riemannian metric g on G, such that the trajectories of ξ be geodesics of g, if and only if*

$$\xi \notin L_\xi\big(L(G)\big). \tag{1}$$

Theorem 3. *Let ξ be a non-vanishing left invariant vector field on a Lie group G.*

(i) *If*

$$\xi \in L_\xi\big(L(G)\big) \tag{2}$$

then there exists a left invariant Lorentzian metric g on G, such that ξ be light-like and the trajectories of ξ be geodesics of g.

(ii) *Suppose* $\dim L_\xi(L(G)) = \dim G - 1$. *If there exists a left invariant Lorentzian metric g on G, such that the trajectories of ξ be geodesics of g and ξ be light-like, then the relation* (2) *holds.*

Theorems 2 and 3 are refinements of the first, allowed by the special invariant structures on G. In addition to the light our results throw on the 'geodesibility' of vector fields on Lie groups, there is another by-product, maybe more important: the different behaviours of the Lie groups, emerging from Theorems 2 and 3, 'classify' them in 7 types (Section 4).

2. Proof of Theorem 1

Let ξ be a left invariant vector field on an n-dimensional Lie group G. For $\xi = 0$, every trajectory is a point, so they may be interpreted as stationary geodesics for any Riemannian metric on G. In the following, we suppose $\xi \neq 0$. Denote by L_ξ the Lie derivative operator associated to ξ.

We look for a Riemannian metric g on G (not necessarily left invariant) such that $\nabla_\xi \xi = 0$, where the Levi-Civita connection ∇ is given by the Koszul's formula [5]

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g\big(X,[Y,Z]\big) + g\big(Y,[Z,X]\big) + g\big(Z,[X,Y]\big)$$

for every X, Y, Z in the Lie algebra $L(G)$. Replacing $X = Y := \xi$, the 'geodesic' property of ξ becomes, for every $Z \in L(G)$,

$$Zg(\xi,\xi) = 2\xi g(\xi, Z) - 2g\big(\xi, [\xi, Z]\big). \tag{3}$$

In order to prove a Riemannian metric g with the property (1) exists, we proceed by two steps: first, we construct a differential 1-form ω on G, such that $\omega(\xi) = 1$ and

$$2L_\xi \omega = d\big(\omega(\xi)\big) = 0. \tag{4}$$

Then, we construct a Riemannian metric g with the property $g(\xi,\cdot) = \omega$. Due to (4), the metric g satisfies the relation (3), thus ending the proof.

Step I. Consider an adapted basis of the Lie algebra of the form $\{e_1 = \xi, e_2, \ldots, e_n\}$ and a dual basis of 1-forms $\{\lambda_1, \ldots, \lambda_n\}$. An arbitrary 1-form ω writes $\omega = \omega_i \lambda^i$ (with the indice i summing from 1 to n), where the coefficients ω_i are differentiable functions on G. We look for ω satisfying $\omega_1 = 1$ and the relation (4); hence, its remaining coefficients must be solutions for the following system of PDEs:

$$2e_1(\omega_j) = c^k_{1j}\omega_k, \quad \text{for } j = \overline{2,n}$$

where c^a_{bc} are the structure constants of $L(G)$ with respect to the previous adapted basis. This linear system with constant coefficients admits some global solutions on G, denoted $\omega_2, \ldots, \omega_n$. We proved that a 1-form ω exists, with the properties required by (4).

Step II. We define $g = \omega \otimes \omega + \lambda^2 \otimes \lambda^2 + \cdots + \lambda^n \otimes \lambda^n$. Obviously, g is semi-positively defined. A short calculation shows that g is non-degenerate and that ω is the 1-form associated to ξ.

3. Proof of Theorems 2 and 3

Consider g a left invariant Riemannian metric on the Lie group G; denote by ∇ its Levi-Civita connection and $n = \dim G$. Suppose $\nabla_\xi \xi = 0$. From Koszul's formula we derive

$$g(\nabla_\xi \xi, Z) = -g\big(\xi, [\xi, Z]\big) \tag{5}$$

for any $Z \in L(G)$. So, ξ is orthogonal to $L_\xi(L(G))$; as $\xi \neq 0$, it follows $\xi \notin L_\xi(L(G))$.

Conversely, suppose $\xi \notin L_\xi(L(G))$. As $L_\xi : L(G) \to L(G)$ is not surjective (because $L_\xi(\xi) = 0$), we have $\dim L_\xi(L(G)) \leqslant n-1$. Then, there exists a left invariant one-form ω on G such that $\omega(L_\xi(L(G))) = 0$ and $\omega(\xi) = 1$. By means of an adapted (orthonormal) basis $\{e_1 := \xi, e_2, \ldots, e_n\}$ in $L(G)$ and a dual one $\{\omega^1 := \omega, \omega^2, \ldots, \omega^n\}$ in $L(G)^*$, we construct a positively defined bilinear form g on $L(G)$, such that $g(\xi, \cdot) = \omega$. Through left translations we extend g to a left invariant Riemannian metric on G (denoted also by g). Applying Koszul's formula (similar to (5)), we obtain

$$g(\nabla_\xi \xi, Z) = \omega\big(L_\xi(Z)\big)$$

for every $Z \in L(G)$, so $\nabla_\xi \xi = 0$. We showed ξ is a geodesic vector field, so Theorem 2 is proved.

For Theorem 3(i) suppose $\xi \in L_\xi(L(G))$. Denote by $k = \dim L_\xi(L(G))$. We may suppose $k \neq 0$ (if not, the adjoint operator associated with ξ has trivial image, so (5) is obviously satisfied).

We choose a basis $\{e_1, e_2, \ldots, e_n\}$ in $L(G)$ such that $\xi = e_1 - e_2$ and $L_\xi(L(G)) \subset \mathrm{sp}\{\xi, e_3, \ldots, e_n\}$. Then, there exists a bilinear Lorentzian scalar product on $L(G)$ such that $\{e_1, e_2, \ldots, e_n\}$ be orthogonal (with $g(e_1, e_1) = -1$, $g(e_i, e_i) = 1$ for $i > 1$). By left translations, we construct a left invariant Lorentzian metric (denoted also by g), with the Levi-Civita connection denoted by ∇. We obtain $\xi \perp L_\xi(L(G))$ and, via (5), we deduce $\nabla_\xi \xi = 0$. Obviously, $g(\xi, \xi) = 0$, so ξ is light-like.

Suppose $k = n-1$ and consider a left invariant Lorentzian metric g on G such that $\nabla_\xi \xi = 0$ and ξ is light-like with respect to g (i.e. it is non-null and $g(\xi, \xi) = 0$). From (5) it follows $\xi \perp L_\xi(L(G))$. Because $\dim L_\xi(L(G)) = n-1$, a standard argument from Lorentzian vector spaces theory implies $\xi \in L_\xi(L(G))$. Theorem 3(ii) is completely proved.

4. Comments

Remark. In general, the assertion of Theorem 3(ii) does not admit extensions for arbitrary dimensions of $L_\xi(L(G))$.

Remark. In the following, all the left invariant vector fields are supposed to be non-null. Elementary combinations of logical quantifiers in the relations (1) and (2) allow the 'classification' of the Lie groups in 7 types:
I. *Those in which* every *left invariant vector field has the property* (1).

The property (1) means that all the adjoint operators on $L(G)$ cannot have non-null real eigenvalues. This happens, for example, on Lie groups admitting a bi-invariant Riemannian metric (and hence all the adjoint operators are skew-symmetric) [4]. The property, however, is not limited to such Lie groups, as the example of the Heisenberg group shows.

Lie groups admitting bi-invariant Lorentzian metrics seem to adopt the same behaviour. For example, the oscillator groups G_λ [2] are the only simply connected, solvable, non-commutative Lie groups to admit bi-invariant Lorentzian metrics. Their Lie algebra admits a basis $\{e_{-1}, e_0, e_1, \ldots, e_n, \tilde{e}_1, \ldots, \tilde{e}_n\}$ such that the only non-null brackets (modulo antisymmetry) are

$$[e_{-1}, e_i] = \lambda_i \tilde{e}_i, \quad [e_i, \tilde{e}_j] = e_0, \quad [e_{-1}, \tilde{e}_i] = -\lambda_i e_i$$

with positive constants $\lambda_1, \ldots, \lambda_n$. We easily check that the condition (1) is fulfilled.
II. *Those in which* every $\xi \in L(G)$ *has the property* (2) *and* $\dim L_\xi(L(G)) = n-1$;
III. *Those in which* every $\xi \in L(G)$ *has the property* (2) *and* $\dim L_\xi(L(G)) \neq n-1$;
IV. *Those in which* every *left invariant vector field* ξ *has the property* (2), *there exist* α, $\beta \in L(G)$, *with* $\dim L_\alpha(L(G)) = n-1$ *and* $\dim L_\beta(L(G)) \neq n-1$;

So, every left invariant vector field is eigenvector with non-null real eigenvalue for some adjoint operator on $L(G)$.

Such a group must be a perfect group (or Cain group in Postnikov's terminology [7]), i.e. must satisfy $[L(G), L(G)] = L(G)$. Hence, the radical and the nilradical coincide. Via the Levi–Malcev theorem [6], the Lie algebra $L(G)$ splits in a semi-direct product of its nilradical and a semi-simple subalgebra. The group cannot be nilpotent.

V. *Those in which* there exist *left invariant vector fields* ξ *having the property* (1), $\eta, \theta \in L(G)$ *with the property* (2) *and* $\dim L_\eta(L(G)) = n-1$, $\dim L_\theta(L(G)) \neq n-1$;

VI. *Those in which* there exist *left invariant vector fields* ξ *having the property* (1), $\eta \in L(G)$ *with the property* (2) *and* $\dim L_\eta(L(G)) = n-1$, *and* there does not exist $\theta \in L(G)$ *with the property* (2) *and* $\dim L_\theta(L(G)) \neq n-1$;

VII. *Those in which* there exist *left invariant vector fields* ξ *having the property* (1), $\eta \in L(G)$ *with the property* (2) *and* $\dim L_\eta(L(G)) \neq n-1$, *and* there does not exist $\theta \in L(G)$ *with the property* (2) *and* $\dim L_\theta(L(G)) = n-1$.

The Lie groups in the class σ have the following characteristic property [4]: for every left invariant vector fields X and Y, their bracket belongs to the subspace generated by X and Y. These solvable Lie groups admit vector fields satisfying both (1) and (2). In particular, the non-Abelian 2-dimensional Lie algebra belongs to the family VI.

This classification is not up to isomorphism; however, isomorphic Lie groups belong to the same class. Possible refinements would take into account the properties of left invariant vector fields ξ, with $\dim L_\xi(L(G)) \leqslant n-2$.

References

[1] V. Arnold, Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits, Ann. Inst. Fourier 16 (1966) 316–361.
[2] A.S. Bromberg, B.A. Medina, Geometry of oscillator groups and locally symmetric manifolds, Geom. Dedicata 106 (2004) 97–111.
[3] B. Khesin, Topological fluid dynamics, Notices Amer. Math. Soc. 52 (1) (2005) 9–19.
[4] J. Milnor, Curvatures of left invariant metrics on Lie groups, Adv. Math. 21 (1976) 293–329.
[5] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press, NY, 1983.
[6] A.L. Onishchik, E.B. Vinberg, Lie Groups and Lie Algebras III, Encyclopaedia Math. Sci., vol. 41, Springer, 1994.
[7] M. Postnikov, Lie Groups and Lie Algebras, English ed., Mir Publ., 1986.