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Mathematical Analysis

The Birkhoff decomposition in groups of formal diffeomorphisms

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Abstract

Let G_∞ be the group of one parameter identity-tangent diffeomorphisms on the line whose coefficients are formal Laurent series in the parameter ε with a pole of finite order at 0. It is well-known that the Birkhoff decomposition can be defined in such a group. We investigate the stability of the Birkhoff decomposition in subgroups of G_∞ and give a formula for this decomposition. As proven by A. Connes and D. Kreimer, the Birkhoff decomposition is related to renormalization in quantum field theory and we give an application of our results in the last section. ***To cite this article: F. Menous, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

La décomposition de Birkhoff dans certains groupes de difféomorphismes formels. Soit G_∞ le groupe des difféomorphismes formels en une variable, à un paramètre, tangents à l'identité, dont les coefficients sont des séries de Laurent formelles en le paramètre ε ayant un pôle d'ordre fini en 0. On peut définir la décomposition de Birkhoff dans un tel groupe. Nous étudions la stabilité par décomposition de Birkhoff de certains sous-groupes de G_∞ et donnons une formule pour cette décomposition. D'après les résultats de A. Connes et D. Kreimer, la décomposition de Birkhoff est liée à la théorie de la renormalisation et nous donnons une application de nos résultats dans la dernière section. ***Pour citer cet article : F. Menous, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction

Let $\mathcal{A}$ the ring of formal Laurent series with a pole of finite order:

$$\forall f \in \mathcal{A},\ \exists \sigma_0 \in \mathbb{Z}; \quad f(\varepsilon) = \sum_{\sigma \geqslant \sigma_0} f_\sigma \varepsilon^\sigma \quad (f_\sigma \in \mathbb{C}).$$

We shall also define two subalgebras of $\mathcal{A}$: $\mathcal{A}_- = \varepsilon^{-1}\mathbb{C}[\varepsilon^{-1}]$ and $\mathcal{A}_+ = \mathbb{C}[[\varepsilon]]$.

Let us now consider the set

$$G_\infty = \left\{\varphi(x,\varepsilon) \in x + x^2\mathcal{A}[[x]]\right\}.$$

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The set G_∞ is a group for the x-composition defined on $G_\infty \times G_\infty$ by

$$\forall(\varphi,\psi) \in G_\infty \times G_\infty, \quad (\varphi \circ \psi)(x,\varepsilon) = \varphi\big(\psi(x,\varepsilon),\varepsilon\big) \tag{1}$$

and, if

$$\begin{aligned} G_\infty^+ &= G_\infty \cap \big\{x + x^2\mathcal{A}_+[[x]]\big\} = G_\infty \cap \mathbb{C}[[x,\varepsilon]], \\ G_\infty^- &= G_\infty \cap \big\{x + x^2\mathcal{A}_-[[x]]\big\} = G_\infty \cap \big\{x + x^2\varepsilon^{-1}\mathbb{C}\big[\varepsilon^{-1}\big][[x]]\big\}. \end{aligned} \tag{2}$$

It is obvious to check that $(G_\infty^+,\circ)$ and $(G_\infty^-,\circ)$ are subgroups of $(G_\infty,\circ)$ and, for any $\varphi \in G_\infty$, there exists a unique pair $B(\varphi) \in (\varphi_-,\varphi_+) \in G_\infty^- \times G_\infty^+$ such that $\varphi \circ \varphi_- = \varphi_+$ and B is called the Birkhoff decomposition of φ (see [1]). The aim of this Note is to give a formula for the Birkhoff decomposition and to exhibit subgroups of G_∞ that are stable under this decomposition. In other words, we define some groups $G \subset G_\infty$ such that, if

$$\begin{aligned} G^+ &= G \cap \big\{x + x^2\mathcal{A}_+[[x]]\big\} = G \cap \mathbb{C}[[x,\varepsilon]], \\ G^- &= G \cap \big\{x + x^2\mathcal{A}_-[[x]]\big\} = G \cap \big\{x + x^2\varepsilon^{-1}\mathbb{C}\big[\varepsilon^{-1}\big][[x]]\big\} \end{aligned} \tag{3}$$

once again $(G^+,\circ)$ and $(G^-,\circ)$ are subgroups of $(G,\circ)$ and the Birkhoff decomposition is stable in G:

$$\forall\varphi \in G, \quad B(\varphi) \in (\varphi_-,\varphi_+) \in G^- \times G^+. \tag{4}$$

2. Subgroups of G_∞

Definition 2.1. For $N \geqslant 0$, let

$$G_N = \big\{\varphi \in G_\infty;\ \varepsilon^{-N}\varphi\big(\varepsilon^N x,\varepsilon\big) \in x + x^2\mathbb{C}[[x,\varepsilon]]\big\}.$$

The series in these sets can be seen as formal identity-tangent diffeomorphisms in x with coefficients in $\mathcal{A}$ and if

$$H_N = \big\{\eta = (n,\sigma) \in \mathbb{N}^* \times \mathbb{Z};\ n \geqslant 1,\ \sigma \geqslant -Nn\big\} \tag{5}$$

then, for $\varphi \in G_N$,

$$\varphi(x,\varepsilon) = x + \sum_{\eta \in H_N} a_\eta x^{n+1}\varepsilon^\sigma \quad (\forall\eta \in H_N,\ a_\eta \in \mathbb{C}). \tag{6}$$

It is easy to check that

$$G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_\infty \tag{7}$$

and

Theorem 2.2. *For $N \in \mathbb{N}$, G_N is a subgroup of G_∞*

The proof of this theorem is based on the composition Faà di Bruno formula. We will now define new subgroups for which the growth of the coefficients of the formal diffeomorphisms can be controlled. In a sense, these subgroups look like 'quasianalytic classes' or 'Gevrey classes'.

Definition 2.3. Let $M_* = \{M_\eta \in \mathbb{R}^{+*},\ \eta \in H_0\}$ a set of positive numbers with such that

$$\forall(\eta_1,\eta_2) \in H_0^2, \quad M_{\eta_1}M_{\eta_2} \leqslant M_{\eta_1+\eta_2} \tag{8}$$

then, $G_N(M_*)$ is the subset of G_N such that, if $\varphi \in G_N(M_*)$, then

$$\varepsilon^{-N}\varphi\big(\varepsilon^N x,\varepsilon\big) = x + \sum_{\eta \in H_0} a_\eta x^{n+1}\varepsilon^\sigma$$

and there exists $A > 0$ such that

$$\forall\eta \in H_0, \quad |a_\eta| \leqslant A^{n+\sigma}M_\eta. \tag{9}$$

This definition makes sense in terms of groups:

Theorem 2.4. *For $N \in \mathbb{N}$, and M_* a set with the property* (8), $(G_N(M_*), \circ)$ *is a subgroup of* $(G_N, \circ)$.

Once again the proof is based on the Faà di Bruno formula and on majorant series.

3. Stability of the Birkhoff decomposition in G_N and $G_N(M_*)$

Let $G = G_N$ and $H = H_N$ ($N \in \mathbb{N}$). Any given $\psi \in G$ defines a substitution automorphism F_ψ on $\mathcal{A}[[x]]$:

$$\forall f \in \mathcal{A}[[x]], \quad (F_\psi.f)\big(x,\varepsilon\big) = f\big(\psi(x,\varepsilon),\varepsilon\big) \tag{10}$$

and $F_\psi.x = \psi(x,\varepsilon)$. For a given $\varphi \in G$: $\varphi(x,\varepsilon) = x + \sum_{\eta\in H} a_\eta \varepsilon^\sigma x^{n+1}$, the operator F_φ can be written:

$$F_\varphi = \mathrm{Id} + \sum_{\eta\in H} \varepsilon^\sigma \mathbb{D}_\eta,$$

where, if $\eta = (n,\sigma)$, $\mathbb{D}_\eta$ is a differential operator in x such that $\mathbb{D}_\eta.x^k = \beta_{\eta,k} x^{n+k}$ where $k \geqslant 0$ and $\beta_{\eta,k} \in \mathbb{C}$. Using these elementary operators, one can find a formula for the Birkhoff decomposition.

Theorem 3.1. *Let $\varphi \in G$, then its Birkhoff decomposition $B(\varphi) = (\varphi_-, \varphi_+)$ ($\varphi \circ \varphi_- = \varphi_+$) is in $G^- \times G^+$ and*

$$\begin{aligned}\varphi_-(x,\varepsilon) &= x + \sum_{s\geqslant 1}\sum_{(\eta_1,\dots,\eta_s)\in H^s} U^{\eta_1,\dots,\eta_s}\mathbb{D}_{\eta_s}\cdots\mathbb{D}_{\eta_1}.x,\\ \varphi_+(x,\varepsilon) &= x + \sum_{s\geqslant 1}\sum_{(\eta_1,\dots,\eta_s)\in H^s} V^{\eta_1,\dots,\eta_s}\mathbb{D}_{\eta_s}\cdots\mathbb{D}_{\eta_1}.x,\end{aligned} \tag{11}$$

where the coefficients $U^\bullet$ and $V^\bullet$ are such that, for any sequence $(\eta_1,\dots,\eta_s) \in H^s$:

$$\begin{aligned}U^{\eta_1,\dots,\eta_s} &= (-1)^s \rho_-(\sigma_1+\cdots+\sigma_s)\rho_-(\sigma_2+\cdots+\sigma_s)\cdots\rho_-(\sigma_s)\varepsilon^{\sigma_1+\cdots+\sigma_s},\\ V^{\eta_1,\dots,\eta_s} &= (-1)^{s-1} \rho_+(\sigma_1+\cdots+\sigma_s)\rho_-(\sigma_2+\cdots+\sigma_s)\cdots\rho_-(\sigma_s)\varepsilon^{\sigma_1+\cdots+\sigma_s},\end{aligned} \tag{12}$$

where $\rho_-(\sigma) = 1$ (resp. 0) if $\sigma < 0$ (resp. $\sigma \geqslant 0$) and $\rho_+ = 1 - \rho_-$.

This formula is based on elementary computations on substitution automorphism and it is easy to check that this also proves that the subgroups G_N are stable under Birkhoff decomposition.

With the help of the formula (11), one can also prove that the subgroups $G_N(M_*)$ are stable under Birkhoff decomposition. This seems difficult to prove directly on formula (11) by a term-by-term majoration of the coefficients because many terms contribute to the same power of x and there exists some tricky compensations between these coefficients. Nevertheless, the difficulty can be circumvented by a rearrangement of the terms of the series in (11) using the arborification–coarborification process defined by J. Ecalle (see [2], Section 4). The key idea for this process is to expand each operator $\mathbb{D}_{\eta_1,\dots,\eta_s} = \mathbb{D}_{\eta_s}\cdots\mathbb{D}_{\eta_1}$, indexed by a fully ordered sequence ($1 < \cdots < s$), as a sum of operators indexed by sequences $(\eta_1,\dots,\eta_s)^<$ equipped with a partial order on the set $\{1,\dots,s\}$, using the Leibniz rules for differential operators. This induces a dual operation on the coefficients $U^\bullet$ or $V^\bullet$ such that the new series defines the same substitution automorphism and delivers the right growth of the coefficients by a term-by-term majoration.

We finally have a great family of subgroups of G_∞ that are stable under Birkhoff decomposition and it is important to notice that this may be applied to renormalization theory.

4. Birkhoff decomposition and renormalization

In quantum field theory, it was proven by A. Connes and D. Kreimer that, after dimensional regularization, the unrenormalized effective coupling constants are the image by a formal identity-tangent diffeomorphism of the coupling constants of the theory. Moreover, the coefficients of this diffeomorphism are Laurent series in the parameter ε associated to the dimensional regularization and the Birkhoff decomposition of this diffeomorphism gives directly the bare coupling constants and the renormalized coupling constants. As proven in [1], in the case of the massless ϕ_6^3 theory, there is an unrenormalized effective coupling constant φ that appears to belong to G_∞ and then, if $B(\varphi) = (\varphi_-, \varphi_+)$, then $\varphi_+(x, 0)$ is the renormalized effective constant and $\varphi_-(x, \varepsilon)$ is the bare coupling constant. This result motivated our study. In fact, the case $\varphi \in G_\infty$ is simple and there seems to be no need for more accurate results on the Birkhoff decomposition.

However, following the results of [4] and [3] for the ϕ_4^4 theory, it seems reasonable to conjecture that, in the massless ϕ_6^3 theory, if

$$\varphi(x, \varepsilon) = x + \sum_{n\geqslant 1} \varphi_n(\varepsilon) x^{n+1}$$

then:

(1) there exists $r > 0$ such that, for $n \geqslant 1$, $\varepsilon^n \varphi_n(\varepsilon)$ is analytic and bounded in the disc of center 0 and radius r/n $(D(0, r/n))$;
(2) there exists $\alpha, A > 0$ such that, for $n \geqslant 1$, $\|\varepsilon^n \varphi_n(\varepsilon)\|_{D(0,r/n)} \leqslant A^n.n^{\alpha n}$;
(3) for $n \geqslant 1$, φ_n is analytically continuable to $\mathbb{C}/\mathbb{R} \cup D(0, r/n)$.

The properties (1) and (2) show that $\varphi \in G_1$. Moreover, if

$$P_1 \varepsilon^{-1} \varphi(\varepsilon x, \varepsilon) = x + \sum_{\eta\in H_0} a_\eta x^{n+1} \varepsilon^\sigma$$

then

$$\forall \eta \in H_0, \quad |a_\eta| \leqslant r^{-\sigma} A^n.n^\sigma.n^{\alpha n}. \tag{13}$$

This means that $\varphi \in G_1(M_*)$ where, for $\eta \in H_0$, $M_\eta = n^\sigma.n^{\alpha n}$ (note that $M_{\eta_1} M_{\eta_2} \leqslant M_{\eta_1+\eta_2}$).

This simply means that φ_- and φ_+ are in $G_1(M_*)$, which finally means that φ_- and φ_+ have the properties (1) and (2). It is obvious to see that φ_- has also the property (3) because,

$$\forall n \geqslant 1, \quad \varphi_{-,n}(\varepsilon) \in \varepsilon^{-1}\mathbb{C}\big[\varepsilon^{-1}\big]$$

and, as $\varphi \circ \varphi_- = \varphi_+$, it is easy to check that φ_+ has also the property (3).

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