Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
C. R. Acad. Sci. Paris, Ser. I 342 (2006) 711–716
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Mathematical Problems in Mechanics

Uniqueness and continuous dependence on the initial data for a class of non-linear shallow shell problems

John Cagnol [a,1], Irena Lasiecka [b,1], Catherine Lebiedzik [c], Richard Marchand [d]

[a] *Pôle universitaire Leonard-de-Vinci, ESILV, DER CS, 92916 Paris La Défense cedex, France*
[b] *University of Virginia, Department of Mathematics, Kerchof Hall, P.O. Box 400137, Charlottesville, VA 22904, USA*
[c] *Wayne State University, Department of Mathematics, 656 W. Kirby, Room 1150, Detroit, MI 48202, USA*
[d] *Slippery Rock University, Department of Mathematics, 229 Vincent Science Hall, Slippery Rock, PA 16057, USA*

Received 31 January 2006; accepted 10 February 2006

Available online 24 March 2006

Presented by Philippe G. Ciarlet

Abstract

This note is concerned with the non-linear shallow shell model introduced in 1966 by W.T. Koiter, and later studied by M. Bernadou and J.T. Oden. We show the uniqueness of the solution to the dynamical model and that this solution is continuous with respect to the initial data. ***To cite this article: J. Cagnol et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Théorème d'unicité et de dépendence continue des solutions par rapport aux conditions initiales pour une classe de problèmes non linéaires de coques peu profondes. Dans cette Note, nous nous intéressons au modèle introduit en 1966 par W.T. Koiter, puis étudié par M. Bernadou et J.T. Oden. Nous démontrons l'unicité de la solution du modèle dynamique et que cette solution est continue par rapport aux conditions initiales. ***Pour citer cet article : J. Cagnol et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Dans [9, Section 11], W.T. Koiter introduit un modèle non-linéaire de coques peu profondes. Ce modèle a été discuté dans [2] et résumé dans [1, Section 7.3]. M. Bernadou et J.T. Oden ont démontré dans [2] que la solution du problème statique existe et qu'elle est unique lorsque la charge est suffisament petite. Dans cette note, nous nous intéressons au modèle dynamique. On fait reference à [5] pour la théorie des coques. L'existence de solution faibles résulte de la méthode de Faedo–Galerkin (cf. [10]), toutefois à notre connaissance, l'unicité était ouverte. En effet, il n'y a pas de propriété de régularisation comme on pourrait en voir dans les problèmes paraboliques. Dans le cas

E-mail addresses: John.Cagnol@devinci.fr (J. Cagnol), il2v@virginia.edu (I. Lasiecka), kate@math.wayne.edu (C. Lebiedzik), richard.marchand@sru.edu (R. Marchand).

[1] Research supported by NSF–INRIA grant number NSF-Int-0226961.

1631-073X/$ – see front matter © 2006 Académie des sciences. Published by Elsevier SAS. All rights reserved.
doi:10.1016/j.crma.2006.02.034

qui nous intéresse, les termes non-linéaires n'ont pas de propriété de conservation d'énergie, ils sont typiquement non bornés dans l'espace de l'énérgie et une propriété qui peut être formellement calculée, peut ne pas être rigoureuse par absence des bornes a priori. Le but de cette Note est d'établir l'unicité de la solution et la continuité de cette solution par rapport aux conditions initiales. Nous utiliserons les méthodes de géométrie intrinsèque de Michel Delfour et Jean-Paul Zolésio, qui utilisent la fonction distance orientée pour décrire la géométrie (cf. [7,6]).

Soit $\mathcal{S}_h$ une coque peu profonde d'épaisseur $h > 0$ encastrée sur le bord, constituée d'un matériau isotrope et homogène. On suppose que l'hypothèse de Kirchoff–Love est satisfaite et que les deflections sont suffisament petites pour entrer dans les conditions décrites par Koiter dans [9, Section 11]. La modélisation du déplacement de la surface moyenne de la coque $\mathbf{e} = (e_\Gamma, w)$ conduit à l'étude de (2).

Théorème 1 *(Unicité et Régularité). Considérons l'équation* (2) *et les espaces* V^m *définis par* (1)*. Si* $\mathbf{e}(0) \in V^1(\Gamma)$ *et* $\partial_t \mathbf{e}(0) \in V^0(\Gamma)$ *alors il existe une unique solution faible* $\mathbf{e}$ *telle que* $\mathbf{e} \in C([0,\tau]; V^1(\Gamma))$ *et* $\partial_t \mathbf{e} \in C([0,\tau]; V^0(\Gamma))$ *(énergie finie). Si en outre* $\mathbf{e}(0) \in V^2(\Gamma)$ *et* $\partial_t \mathbf{e}(0) \in V^1(\Gamma)$ *alors il existe une solution unique globale régulière* $\mathbf{e} \in C([0,\tau]; V^2(\Gamma)) \cap C^1([0,\tau]; V^1(\Gamma)) \cap C^2([0,\tau]; V^0(\Gamma))$ *où* $\tau > 0$ *est arbitraire.*

Théorème 2 *(Dépendance continue par rapport aux conditions initiales). Considérons l'éq.* (2) *et les espaces* V^m *définis par* (1)*. La solution faible* (2) *dépendent continûment des conditions initiales dans la norme de l'énergie : pour tout* $\tau > 0$ *et toutes suites de conditions initiales telles que* $\mathbf{e}^n(0) \to \mathbf{e}_0$ *dans* $V^1(\Gamma)$ *et* $\partial_t \mathbf{e}^n(0) \to \mathbf{e}_1$ *dans* $V^0(\Gamma)$*. Les solutions correspondantes* $\mathbf{e}^n(t)$ *et* $\mathbf{e}(t)$ *appartiennent à* $C([0,\tau]; V^1(\Gamma))$ *et satisfont* $\mathbf{e} \to \mathbf{e}$ *dans* $C([0,\tau]; V^1(\Gamma))$ *et* $\partial_t \mathbf{e} \to \partial_t \mathbf{e}$ *dans* $C([0,\tau]; V^0(\Gamma))$.

Cette Note se décompose en une Section 1 dans laquelle on obtient le modèle en formulation intrinsèque (cf. [6]), les Sections 2 et 3 dans lesquelles on démontre le Théorème 1 et la Section 4 qui est consacrée à la démonstration du Théorème 2.

1. Introduction and modeling

In [9, Section 11], W.T. Koiter introduces a shallow shell model for *small finite deflections*, characterized by small displacement gradients and by rotations whose squares do not exceed the middle surface strain in order of magnitude. This corresponds to the *medium bending case* in the sense of Naghdi (see [11]) and to the *approximation of small strains and moderately small rotations* in the sense of Lyell and Sanders (see [12]). This case leads to geometrically non-linear model discussed in [2] and summarized in [1, Section 7.3]. In [2], M. Bernadou and J.T. Oden have proved the solution to static model exists and that it is unique provided the load is sufficiently small. See [5] for the linear and non-linear theories of shells.

In this Note we are concerned with the dynamic model. Existence of weak solutions for such model follows from the Faedo–Galerkin method [10]. However, the uniqueness of the solution and the continuous dependence of solutions with respect to the initial data is an open problem. The issue of uniqueness is subtle, as there is no inherent "smoothing" property such as one would see with a parabolic system. In addition, the problem is non-monotone, the non-linear terms are not locally Lipschitz and are not in the energy space, therefore the standard methods fail. The aim of the present note is to establish such uniqueness, to prove the solution is continuous with respect to the initial data as well as the existence of strong solutions.

We shall use the intrinsic geometry methods of Michel Delfour and Jean-Paul Zolésio, which relies on the oriented distance function to describe the geometry [6,7]. The shell is described in terms of tangential differential operators which are defined by means of the oriented boundary distance function in $\mathbb{R}^3$. Sobolev spaces, Green's formula, and key inequalities such as Poincaré and Korn's inequalities are all well-defined.

The displacement of the shell T and the displacement of the mid-surface of this shell $\mathbf{e}$ are related by the Kirchoff–Love hypothesis. We refer to [3] for the relation between T and $\mathbf{e}$. The kinetic energy of the system is given in [3, Proposition 8]. The key hypothesis, and main difference with [3], is the strain–displacement relations

$$\varepsilon(T) = \frac{1}{2}({}^*DT + DT) + \frac{1}{2}{}^*DT\,DT$$

and the fact that the shell is assumed shallow, which eliminates a third-order coupling between normal and tangential displacements. Several terms of the non-linear term ${}^*DT\,DT$ will be neglected, this will especially be the case of all the terms involving ${}^*D_\Gamma e_\Gamma$ that appear in ${}^*DT\,DT$.

Let w be a scalar and u a vector. Following the definitions introduced in [3], let $G_\Gamma w = \frac{1}{2}((\nabla b \otimes \nabla_\Gamma w)D^2 b + D^2 b(\nabla_\Gamma w \otimes \nabla b))$ be a 1st-order tangential operator, $V_\Gamma u = \frac{1}{2}((D^2 bu) \otimes \nabla b + \nabla b \otimes (D^2 bu))$ be a zero-order tangential operator and $S_\Gamma w = \frac{1}{2}(D^2_\Gamma w + {}^*D^2_\Gamma w)$ be the symmetrization of the Hessian (which is *not* symmetric in the tangential calculus [6]).

Let $\mathcal{S}_h$ be a shallow shell of thickness $h > 0$ clamped along its edges, made of an isotropic and homogeneous material. We assume the Kirchoff–Love hypothesis and the small finite deflection hypothesis. Let $\gamma = \frac{h^2}{12} > 0$. We denote by λ and μ the Lamé coefficient and $\mathcal{C}(A) = \lambda(\operatorname{tr} A)I + 2\mu A$. We note H the mean curvature, K the Gauss curvature and $k = \lambda H + \mu\sqrt{2H^2 - K}$, which is a positive real number.

Let $\mathbf{e} = e_\Gamma + w\nabla b$ be the displacement of the mid-surface of the shell. Hooke's law gives the elastic energy $\mathcal{E}_p(\mathbf{e})$. We impose the hypothesis of plane stresses and derive the elastic strain tensor. Let us note

$$B_M(\mathbf{e}) = \varepsilon_\Gamma(e_\Gamma) + V_\Gamma e_\Gamma + wD^2 b + \frac{1}{2}\nabla_\Gamma w \otimes \nabla_\Gamma w$$

the membrane strain,

$$B'_M(\mathbf{e}, \hat{\mathbf{e}}) = \varepsilon_\Gamma \widehat{e_\Gamma} + V_\Gamma \widehat{e_\Gamma} + \widehat{w} D^2 b + \frac{1}{2}\nabla_\Gamma w \otimes \nabla_\Gamma \widehat{w} + \frac{1}{2}\nabla_\Gamma \widehat{w} \otimes \nabla_\Gamma w$$

its gateaux derivative and $B_F(\mathbf{e}) = S_\Gamma w + G_\Gamma w$ the bending (flexural) strain. We have

$$\begin{aligned}\varepsilon(T) &= \big(B_M(\mathbf{e})\big) \circ p + b\big(B_F(\mathbf{e})\big) \circ p \\ &= \left(\varepsilon_\Gamma(e_\Gamma) + wD^2 b + V_\Gamma e_\Gamma + \frac{1}{2}\nabla_\Gamma w \otimes \nabla_\Gamma w\right) \circ p - b(S_\Gamma w + G_\Gamma w) \circ p.\end{aligned}$$

This new form of $\varepsilon(T)$ is in line with [1, p. 38, Section 7.3]. Then $\mathcal{E}_p$ is computed as a function of e_Γ, w and the geometrical properties of the shell: b, h, H and K. Hamilton's principle leads to the following modeling equation (2), which motivates its study.

Definition 1 *(Weak Solutions).* For $m \in \mathbb{N}$, define the spaces V^m as

$$V^m(\Gamma) = \left\{\mathbf{e} \in \big[H^m(\Gamma)\big]^2 \times H^{m+1}(\Gamma) \;\middle|\; e_\Gamma = 0,\ w = 0,\ \frac{\partial}{\partial \nu} w = 0 \text{ on } \partial\Gamma\right\}. \tag{1}$$

Let $\mathbf{e} \in L^\infty([0,\tau], V^1(\Gamma))$ with $\partial_t \mathbf{e} \in L^\infty([0,\tau], V^0(\Gamma))$. We say that $\mathbf{e}$ is a weak solution to the small finite deflections shell problem if and only if

$$\mathfrak{m}(\partial_{tt}\mathbf{e}, \hat{\mathbf{e}}) + \mathfrak{a}(\mathbf{e}, \hat{\mathbf{e}}) - \mathfrak{n}(\mathbf{e}, \hat{\mathbf{e}}) = 0 \quad \forall \hat{\mathbf{e}} \in L^1\big([0,\tau], V^1(\Gamma)\big), \tag{2}$$

where $\mathbf{e}(0) = \mathbf{e}_0$ and $\partial_t \mathbf{e}(0) = \mathbf{e}_1$ and $\mathfrak{m}(\mathbf{e}, \hat{\mathbf{e}})$, $\mathfrak{a}(\mathbf{e}, \hat{\mathbf{e}})$, $\mathfrak{n}(\mathbf{e}, \hat{\mathbf{e}})$ are defined by

$$\begin{aligned}\mathfrak{m}(\mathbf{e}, \hat{\mathbf{e}}) &= -2\rho(\mathbf{e}, \hat{\mathbf{e}})_\Gamma - \gamma\big(D^2 b e_\Gamma - \nabla_\Gamma w, D^2 b\widehat{e_\Gamma} - \nabla_\Gamma \widehat{w}\big)_\Gamma, \\ \mathfrak{a}(\mathbf{e}, \hat{\mathbf{e}}) &= \int_\Gamma \big(\mathcal{C}B_M(\mathbf{e})\big) \cdot\cdot B'_M(\mathbf{e}, \hat{\mathbf{e}}) + \gamma \int_\Gamma \big(\mathcal{C}B_F(\mathbf{e})\big) \cdot\cdot B_F(\hat{\mathbf{e}}), \\ -\mathfrak{n}(\mathbf{e}, \hat{\mathbf{e}}) &= 2\lambda\big(H\|\nabla_\Gamma w\|^2, \widehat{w}\big)_\Gamma + \lambda\big(\|\nabla_\Gamma w\|^2, \operatorname{div}_\Gamma \hat{e}_\Gamma\big)_\Gamma + 4\lambda(Hw\nabla_\Gamma w, \nabla_\Gamma \widehat{w})_\Gamma \\ &\quad + 2\lambda(\operatorname{div}_\Gamma e_\Gamma \nabla_\Gamma w, \nabla_\Gamma \widehat{w})_\Gamma + (\lambda + 2\mu)\big(\|\nabla_\Gamma w\|^2 \nabla_\Gamma w, \nabla_\Gamma \widehat{w}\big)_\Gamma + 2\big(w, \big\langle D^2 b \nabla_\Gamma w, \nabla_\Gamma \widehat{w}\big\rangle\big)_\Gamma \\ &\quad + 2\big(\operatorname{tr}\big(D^2 b(\nabla_\Gamma w \otimes \nabla_\Gamma w)\big), \widehat{w}\big)_\Gamma + \mu \int_\Gamma \operatorname{tr}\big(\varepsilon_\Gamma(\hat{e}_\Gamma)(\nabla_\Gamma w \otimes \nabla_\Gamma w)\big) \\ &\quad + 4\mu \int_\Gamma \operatorname{tr}\big((\nabla_\Gamma w \otimes \nabla_\Gamma \widehat{w})(\varepsilon_\Gamma(e_\Gamma) + V_\Gamma e_\Gamma)\big).\end{aligned}$$

2. Existence and uniqueness of weak solutions

Theorem 1 *(Existence and uniqueness of weak solutions). If* $\mathbf{e}(0) \in V^1(\Gamma)$ *and* $\partial_t \mathbf{e}(0) \in V^0(\Gamma)$ *then the weak solution* $\mathbf{e} \in C([0,\tau]; V^1(\Gamma)) \cap C^1([0,\tau]; V^0(\Gamma))$ *exists and is unique, where* τ *is arbitrary.*

2.1. Norm estimates

Let $\mathcal{M}$, $\mathcal{A}$ be the linear operators defined by $(\mathcal{M}\mathbf{e}, \hat{\mathbf{e}}) = \mathfrak{m}(\mathbf{e}, \hat{\mathbf{e}})$ and $(\mathcal{A}\mathbf{e}, \hat{\mathbf{e}}) = \mathfrak{a}(\mathbf{e}, \hat{\mathbf{e}})$, and let

$$\mathcal{H} = \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{D}(\mathcal{M}_\gamma^{1/2}) = (H^1)^2 \times H^2 \times (L^2)^2 \times H^1, \qquad \mathbb{A}: \mathcal{H} \to \mathcal{H},\ \mathbb{A} = \begin{pmatrix} 0 & -I \\ \mathcal{M}_\gamma^{-1}\mathcal{A} & 0 \end{pmatrix}.$$

Let N be the non-linear operator defined by $(N(\mathbf{e}), \hat{\mathbf{e}}) = \mathfrak{n}(\mathbf{e}, \hat{\mathbf{e}})$ and N_Γ, N_n be respectively the vector function and scalar function defined as follows, so we have $N(\mathbf{e}) = N_\Gamma(w) + N_n(e_\Gamma, w)\nabla b$.

$$\begin{aligned} N_\Gamma(w) &= -\lambda \nabla_\Gamma\big(\|\nabla_\Gamma w_\Gamma\|^2\big) + 2\mu \operatorname{div}_\Gamma(\nabla_\Gamma w \otimes \nabla_\Gamma w), \\ N_n(e_\Gamma, w) &= -2\lambda\|\nabla_\Gamma w\|^2 + 4\lambda \operatorname{div}_\Gamma(H w \nabla_\Gamma w) + 2\mu \operatorname{div}_\Gamma\big(w\, D^2 b \nabla_\Gamma w\big) + (\lambda + 2\mu) \operatorname{div}_\Gamma\big(\|\nabla_\Gamma w\|^2 \nabla_\Gamma w\big) \\ &\quad - 2\mu\big\langle D^2 b \nabla_\Gamma w, \nabla_\Gamma w\big\rangle + 2\lambda \operatorname{div}_\Gamma(\operatorname{div}_\Gamma e_\Gamma \nabla_\Gamma w) + 4\mu \operatorname{div}_\Gamma\big((\varepsilon_\Gamma e_\Gamma + V_\Gamma e_\Gamma)\nabla_\Gamma w\big). \end{aligned}$$

Let us denote by $\{\varphi_i\}$ the family of the eigenvectors of the Laplace operator with zero boundary condition defined on S_h and λ_i the eigenvalues corresponding to φ_i. Let P_n be the orthogonal projection on the subspace spanned by the n eigenvectors $\varphi_0, \ldots, \varphi_{n-1}$. If $f \in L^2$ and $g \in H^1$ then, estimate resulting from Sobolev's embedding and the Holder inequality gives [13]:

$$\big\|(P_n f) g\big\|_{L^2} \leqslant C\sqrt{\ln(1+\lambda_n)}\, \|f\|_{L^2} \|g\|_{H^1}. \tag{3}$$

We note $Q_n = I - P_n$. The sequence (λ_n) is increasing and $\lim \lambda_n = +\infty$. First, we establish that for $\epsilon > 0$, $\alpha > 0$ and $f \in H^{\epsilon+\alpha}$, we have $\|Q_n f\|^2_{H^\epsilon} \leqslant \frac{1}{\lambda^\alpha}\|Q_n f\|^2_{H^{\epsilon+\alpha}}$, then, we prove that if $g \in H^1$, we have $\|(Q_n f) g\|^2_{L^2} \leqslant \frac{1}{\lambda_n^\alpha}\|f\|^2_{H^{\epsilon+\alpha}}\|g\|^2_{H^1}$. It follows that

$$\|fg\|^2_{L^2} \leqslant C\ln(1+\lambda_n)\|f\|^2_{L^2} + \frac{1}{\lambda_n^\alpha}\|f\|^2_{H^{\epsilon+\alpha}}\|g\|^2_{H^1}.$$

Corresponding inequalities can be obtained for $\|\langle v_1, v_2\rangle\|^2_{L^2}$ and $\|v_1 \otimes v_2\|^2_{L^2}$. The main issue is that $f \in H^1(\Gamma)$ is not a multiplier of L^2. For this reason we shall use projections in (3) with the constants blowing up (in a controlled manner) when $n \to +\infty$. We define $E(t)$ as the total energy of the system. Formally, one gets $E(t) = E(s)$ for strong solutions and $E(t) \leqslant E(s)$ for weak solutions. However, in forthcoming Lemma 1 we shall establish the validity of the *equality* also for all weak or finite energy solutions.

2.2. Completion of the proof of Theorem 1

Let $\mathbf{e}^1$ and $\mathbf{e}^2$ be two solutions to the variational problem given in Definition 1, with the same initial condition. Let $\tilde{\mathbf{e}} = \mathbf{e}^2 - \mathbf{e}^1$. We prove the following inequalities

$$\big\|\mathcal{M}_\gamma^{1/2}\tilde{\mathbf{e}}\big\|_{L^2} \leqslant \left\|\mathbb{A}^{-1}\begin{pmatrix}\tilde{\mathbf{e}} \\ \partial_t \tilde{\mathbf{e}}\end{pmatrix}\right\|_{\mathcal{H}} \leqslant \int_0^t \left\|\mathbb{A}^{-1}\begin{pmatrix} 0 \\ \mathcal{M}^{-1}[N(\mathbf{e}^2) - N(\mathbf{e}^1)]\end{pmatrix}\right\|_{\mathcal{H}} \mathrm{d}s$$

which lead to

$$\|\tilde{e}_\Gamma\|^2_{L^2} + \|\widetilde{w}\|^2_{H^1} \leqslant t\int_0^t \big(\big\|N_\Gamma\big(w^2\big) - N_\Gamma\big(w^1\big)\big\|^2_{H^{-1}} + \big\|N_n\big(e^2_\Gamma, w^2\big) - N_n\big(e^1_\Gamma, w^1\big)\big\|^2_{H^{-2}}\big)\,\mathrm{d}s.$$

We compute $N_\Gamma(w^2) - N_\Gamma(w^1)$ and $N_n(e^2_\Gamma, w^2) - N_n(e^1_\Gamma, w^1)$. For $\epsilon > 0$ and $\alpha > 0$ such that $\epsilon + \alpha \leqslant 1$, there exists a real C depending only of λ, μ, w^1 and w^2 such that

$$\big\|N_\Gamma\big(w^2\big) - N_\Gamma\big(w^1\big)\big\|^2_{H^{-1}} \leqslant C\ln(1+\lambda_n)\|\widetilde{w}\|^2_{H^1} + \frac{C}{\lambda_n^\alpha}.$$

The estimation on $N_n(e^2_\Gamma, w^2) - N_n(e^1_\Gamma, w^1)$ is slightly more complicated so we introduce

$$X^1(\widetilde{w}, w^1, w^2) = -2\lambda\langle\nabla_\Gamma\widetilde{w}, \nabla_\Gamma(w^2 + w^1)\rangle + 4\lambda \operatorname{div}_\Gamma(Hw^2\nabla_\Gamma\widetilde{w} + H\widetilde{w}\nabla_\Gamma w^1)$$
$$+ 2\mu \operatorname{div}_\Gamma(D^2b(w^2\nabla_\Gamma\widetilde{w} + \widetilde{w}\nabla_\Gamma w^1)) - 2\mu\langle D^2b\nabla_\Gamma\widetilde{w}, \nabla_\Gamma w^2\rangle - 2\mu\langle D^2b\nabla_\Gamma w^1, \nabla_\Gamma\widetilde{w}\rangle$$
$$+ (\lambda+\mu) \operatorname{div}_\Gamma(\nabla_\Gamma\widetilde{w}\|\nabla_\Gamma w^2\|^2 + \nabla_\Gamma w^1\langle\nabla_\Gamma\widetilde{w}, \nabla_\Gamma(w^2 + w^1)\rangle),$$
$$X^2(\widetilde{w}, \tilde{e}_\Gamma, w^1, e^1_\Gamma, w^2, e^2_\Gamma) = 2\lambda \operatorname{div}_\Gamma(\operatorname{div}_\Gamma \tilde{e}_\Gamma\nabla_\Gamma w^2 + \operatorname{div}_\Gamma e^1_\Gamma\nabla_\Gamma\widetilde{w})$$
$$+ 4\mu \operatorname{div}_\Gamma((\varepsilon_\Gamma e^2_\Gamma + V_\Gamma e^2_\Gamma)\nabla_\Gamma\widetilde{w} + (\varepsilon_\Gamma\tilde{e}_\Gamma + V_\Gamma\tilde{e}_\Gamma)\nabla_\Gamma w^1)$$

so we have $N_n(e^2_\Gamma, w^2) - N_n(e^1_\Gamma, w^1) = X^1(\widetilde{w}, w^1, w^2) + X^2(\widetilde{w}, \tilde{e}_\Gamma, w^1, e^1_\Gamma, w^2, e^2_\Gamma)$. Bounding the first term $\|X^1(\widetilde{w}, w^1, w^2)\|_{H^{-2}}$ is an easy task, because all the terms are of lower order. The second term $\|X^2(\widetilde{w}, \tilde{e}_\Gamma, w^1, e^1_\Gamma, w^2, e^2_\Gamma)\|_{H^{-2}}$ is bounded by

$$C(\|\operatorname{div}_\Gamma \tilde{e}_\Gamma\nabla_\Gamma w^2\|_{H^{-1}} + \|\operatorname{div}_\Gamma e^1_\Gamma\nabla_\Gamma\widetilde{w}\|_{H^{-1}} + \|(\varepsilon_\Gamma e^2_\Gamma + V_\Gamma e^2_\Gamma)\nabla_\Gamma\widetilde{w}\|_{H^{-1}}$$
$$+ \|((\varepsilon_\Gamma\tilde{e}_\Gamma) + V_\Gamma\tilde{e}_\Gamma)\nabla_\Gamma w^1)\|_{H^{-1}}).$$

Using the Green's formula for tangential derivatives [6] and a duality argument gives

$$\|\operatorname{div}_\Gamma \tilde{e}_\Gamma\nabla_\Gamma w^2\|_{H^{-1}} \leqslant C\left[\ln(1+\lambda_n)\|\tilde{e}_\Gamma\|_{L_2} + \frac{1}{\lambda_n^\alpha}\|\tilde{e}_\Gamma\|_{H^{\epsilon+\alpha}}\right]\|w^2\|_{H^2}.$$

The second and third terms are more direct, the fourth term requires additional integrations using the Green's formulas

$$\|\operatorname{div}_\Gamma e^1_\Gamma\nabla_\Gamma\widetilde{w}\|_{H^{-1}} \leqslant C\left[\ln(1+\lambda_n)\|\widetilde{w}\|_{H^1} + \frac{1}{\lambda_n^\alpha}\|\widetilde{w}\|_{H^{1+\epsilon+\alpha}}\right]\|e^1_\Gamma\|_{H^1},$$
$$\|(\varepsilon_\Gamma(e^2_\Gamma) + V_\Gamma e^2_\Gamma)\nabla_\Gamma\widetilde{w}\|_{H^{-1}} \leqslant C\left[\ln(1+\lambda_n)\|\widetilde{w}\|_{H^1} + \frac{1}{\lambda_n^\alpha}\|\widetilde{w}\|_{H^{1+\epsilon+\alpha}}\right]\|e^2_\Gamma\|_{H^1},$$
$$\|(\varepsilon_\Gamma\tilde{e}_\Gamma) + V_\Gamma\tilde{e}_\Gamma\nabla_\Gamma w^1\|_{H^{-1}} \leqslant C\left[\ln(1+\lambda_n)\|\tilde{e}_\Gamma\|_{L_2} + \frac{1}{\lambda_n^\alpha}\|\tilde{e}_\Gamma\|_{H^{\epsilon+\alpha}}\right]\|w^1\|_{H^2}.$$

Finally, $\|N_n(e^2_\Gamma, w^2) - N_n(e^1_\Gamma, w^1)\|^2_{H^{-2}} \leqslant C\ln(1+\lambda_n)\|\widetilde{w}\|^2_{H^1} + C\ln(1+\lambda_n)\|\tilde{e}_\Gamma\|^2_{L^2} + \frac{C}{\lambda_n^\alpha}$.

From now on we will consider $\alpha = 1-\epsilon$. We obtain

$$\|\tilde{e}_\Gamma\|^2_{L^2} + \|\widetilde{w}\|^2_{H^1} \leqslant Ct^2\lambda_n^{Ct^2-\alpha/2}$$

where C depends only on $\mathbf{e}^1$, $\mathbf{e}^2$, the Lamé coefficients and the geometric information embedded in b. The sequence (λ_n) tends toward $+\infty$. When $t \in [0, \sqrt{\alpha/(2C)}[$, the right-hand side tends to 0. Consequently $\tilde{\mathbf{e}} = 0$ on $[0, \sqrt{\alpha/(2C)}[$. All of the constants depend on the initial data in finite energy space and on intrinsic geometric constants. This allows to boot-strap the argument obtaining uniqueness for any finite time τ.

3. Regular solutions

Proposition 1 *(Regular solutions). Assume that $\mathbf{e}(0) \in V^2(\Gamma)$ and $\partial_t\mathbf{e}(0) \in V^1(\Gamma)$. Then, there exists a unique global solution $\mathbf{e} \in C([0,\tau]; V^2(\Gamma)) \cap C^1([0,\tau]; V^1(\Gamma)) \cap C^2([0,\tau]; V^0(\Gamma))$ where $\tau > 0$ is arbitrary*

The proof of this proposition is given in [4], it is achieved proving stability in higher norms

4. Continuous dependence with respect to the initial data

Theorem 2 *(Continuous dependence on initial data). Weak solutions to* (2) *depend continuously on the initial data in the finite energy norm. That is to say, for all $\tau > 0$ and all sequences of initial data such that $\mathbf{e}^n(t) \to \mathbf{e}^0$ in $V^1(\Gamma)$ and $\partial_t\mathbf{e}^n(t) \to \mathbf{e}^1$ in $V^0(\Gamma)$, the corresponding solutions $\mathbf{e}^n(t) \in C([0,\tau]; V^1(\Gamma))$ and $\mathbf{e}(t) \in C([0,\tau]; V^1(\Gamma))$ satisfy: $\mathbf{e}^n \to \mathbf{e}$ in $C([0,\tau]; V^1(\Gamma))$ and $\partial_t\mathbf{e}^n \to \partial_t\mathbf{e}$ in $C([0,\tau]; V^0(\Gamma))$. As a consequence we obtain an energy identity for weak solutions.*

The main difficulty in the proof of Theorem 2 is the derivation of the energy *identity* for weak solutions. Formally we can deduce this identity easily by applying test functions $\hat{\mathbf{e}} = \partial_t \mathbf{e}$ in the weak form of the model Proposition 1, but the low regularity of weak solutions prevents this argument from being rigorous.

Lemma 1. *Let $\gamma > 0$. We consider solutions to* (2) *with the a priori regularity*: $\mathbf{e} \in L^\infty([0,\tau]; V^1(\Gamma))$ *and* $\partial_t \mathbf{e} \in L^\infty([0,\tau]; V^0(\Gamma))$ *then,* $0 \leqslant s \leqslant t \leqslant \tau$ *implies* $E(t) = E(s)$.

In order to derive this result, we use as a finite difference approximation of time derivatives as a multiplier. This allows us to derive a relationship between the linear energy terms (where the limit is no problem) and the problematic non-linear terms. Though the product of an H^1 function and a L^2 function does not belong to L^2, the product of two $H^{1/2}$ functions does. Specially formulated finite difference scheme exhibits this better property and allows to pass to the limit on non-linear terms. After some computations and estimations we can pass the limit on the problematic terms and reconstruct the desired energy identity, Lemma 1.

Let us define $\mathbf{x}(t) = (\mathbf{e}, \partial_t \mathbf{e}) = (e_\Gamma(t), w(t), \partial_t e_\Gamma(t), \partial_t w(t))$ where $\mathbf{e} = (e_\Gamma(t), w(t))$ is a weak solution to the model (2) at the time t due to some initial data $\mathbf{e}(0)$. We start with initial data $\mathbf{x}(0) \in \mathcal{H}$ such that $\mathbf{x}^n(0) \to \mathbf{x}(0)$ in $\mathcal{H}$ as $n \to \infty$. Our aim is to prove $\mathbf{x}^n(0) \to \mathbf{x}(0)$ in $C([0,\tau], \mathcal{H})$. We will follow the roadmap presented in [8]. By Lemma 1 we have $\|\mathbf{x}\|_{\mathcal{H}} \leqslant CE(\mathbf{x}(t))$ so that

$$\left|\mathbf{e}^n(t)\right|_{\mathcal{H}} \leqslant C\left(\left|\mathbf{e}^n(0)\right|_{\mathcal{H}}\right) \leqslant C\left(\left|\mathbf{e}(0)\right|_{\mathcal{H}}\right).$$

Hence, on a subsequence denoted by the same index we have $\mathbf{x}^n(t) \to \mathbf{x}^*(t)$ weakly* in $L^\infty(0,\tau;\mathcal{H})$. By using the variational equality together with weak continuity of non-linear terms, we can show that $\mathbf{x}^*(t)$ coincides with a weak solution to (2) due to the initial data $\mathbf{x}(0)$. By Theorem 2, $\mathbf{x}^*(t) = \mathbf{x}(t)$. Hence we have $\mathbf{x}^n(t) \to \mathbf{x}(t)$ weakly* in $L^\infty(0,\tau;\mathcal{H})$. In view of this, to prove the theorem it is enough to show the norm convergence of $\|\mathbf{x}^n(t)\|_{\mathcal{H}} \to |\mathbf{x}(t)|_{\mathcal{H}}$ in $C(0,\tau)$, which we do using the equality in the energy relation from Lemma 1. Combining weak convergence and norm convergence gives the convergence of the corresponding solutions in $C([0,\tau];\mathcal{H})$.

References

[1] M. Bernadou, Méthodes d'Eléments Finis pour les Problèmes de Coques Minces, Masson, Paris, 1994.

[2] M. Bernadou, J.T. Oden, An existence theorem for a class of nonlinear shallow shell problems, J. Math. Pures Appl. (9) 60 (3) (1981) 285–308.

[3] J. Cagnol, I. Lasiecka, C. Lebiedzik, J.-P. Zolésio, Uniform stability in structural acoustic models with flexible curved walls, J. Differential Equations 186 (1) (2002) 88–121.

[4] J. Cagnol, I. Lasiecka, C. Lebiedzik, R. Marchand, Hadamard wellposedness for a class of non-linear shallow shell problems, ESILV, DER-CS RR-28, Pôle Universitaire L. de Vinci, Paris La Défense, France, 2005.

[5] P.G. Ciarlet, Mathematical Elasticity, vol. III: Theory of Shells, Stud. Math. Appl., vol. 29, North-Holland Publishing Co., Amsterdam, 2000.

[6] M.C. Delfour, J.-P. Zolésio, Intrinsic differential geometry and theory of thin shells, in press.

[7] M.C. Delfour, J.-P. Zolésio, Differential equations for linear shells: comparison between intrinsic and classical models, in: Adv. Math. Sci., in: CRM Proc. Lect. Notes, vol. 11, Amer. Math. Soc., 1997, pp. 41–124.

[8] H. Koch, I. Lasiecka, Hadamard well-posedness of weak solutions in nonlinear dynamic elasticity—full von Karman systems, Progr. Nonlinear Differential Equations Appl. 50 (2002) 197–216.

[9] W.T. Koiter, On the nonlinear theory of thin elastic shells. III, Nederl. Akad. Wetensch. Proc. Ser. B 69 (1966) 33–54.

[10] J.-L. Lions, E. Magenes, Problèmes aux limites non homogènes et applications, Dunod, 1968.

[11] P.M. Naghdi, Foundations of elastic shell theory, in: Progr. Solid Mech., vol. IV, North-Holland, Amsterdam, 1963, pp. 1–90.

[12] Jr. Sanders, J. Lyell, Nonlinear theories for thin shells, Quart. Appl. Math. 21 (1963) 21–36.

[13] V.I. Sedenko, On the uniqueness theorem for generalized solutions of initial-boundary problems for the Marguerre–Vlasov of shallow shells with clamped boundary conditions, Appl. Math. Optim. 39 (3) (1999) 309–326.