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Algebraic Geometry

Symplectic resolutions for nilpotent orbits (III)

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Abstract

We prove that two symplectic resolutions of a nilpotent orbit closures in a simple complex Lie algebra of classical type are related by Mukai flops in codimension 2. ***To cite this article: B. Fu, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Résolutions symplectiques pour les orbites nilpotentes (III). Nous montrons que deux résolutions symplectiques d'une adhérence d'orbite nilpotente dans une algèbre de Lie simple complexe classique sont réliées l'une à l'autre par des flops de Mukai en codimension 2. ***Pour citer cet article : B. Fu, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction

A symplectic variety is a complex algebraic variety W, smooth in codimension 1, such that there exists a regular symplectic form on its smooth part which can be extended to a global regular form on any resolution (see [1]). A resolution $\pi : X \to W$ is called *symplectic* if the lifted regular form on X is non-degenerate everywhere. One can show that if W is normal, then a resolution is symplectic if and only if it is crepant.

One way of constructing a symplectic resolution from another is to perform Mukai flops. This process can be described as follows: let W be a symplectic variety and $\pi : X \to W$ a symplectic resolution. Assume that W contains a smooth subvariety Y and $\pi^{-1}(Y)$ contains a smooth subvariety P such that the restriction of π to P makes P a $\mathbb{P}^n$-bundle over Y. If $\operatorname{codim}(P) = n$, then we can blow up X along P and then blow down along the other direction, which gives another (proper) symplectic resolution $\pi^+ : X^+ \to W$, provided that X^+ remains in our category of algebraic varieties. The diagram $X \to W \leftarrow X^+$ is called *Mukai's elementary transformation* (MET for short) over W with center Y. A *MET in codimension 2* is a diagram which becomes a MET after removing subvarieties of codimension greater than 2.

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Conjecture 1.1. [7] *Let W be a symplectic variety which admits two projective symplectic resolutions $\pi : X \to W$ and $\pi^+ : X^+ \to W$. Then the birational map $\phi = (\pi^+)^{-1} \circ \pi : X \dashrightarrow X^+$ is related by a sequence of METs over W in codimension* 2.

Notice that since the two resolutions π, π^+ are both crepant, the birational map ϕ is isomorphic in codimension 1. This conjecture is true for four-dimensional symplectic varieties by the work of Wierzba and Wiśniewski ([9]) (while partial results have been obtained in [2], see also [3]).

Examples of symplectic varieties include: (i) nilpotent orbit closures in a semi-simple complex Lie algebra and (ii) quotient varieties $\mathbb{C}^{2n}/G$ with $G < Sp(2n)$ a finite subgroup. Conjecture 1.1 is verified for case (ii) in [5]. The purpose of this note is to prove the above conjecture for case (i).

Theorem 1.2. *Let $\overline{\mathcal{O}}$ be a nilpotent orbit closure in a complex simple Lie algebra of classical type. Then any two* (*proper*) *symplectic resolutions for $\overline{\mathcal{O}}$ are connected by a sequence of METs over $\overline{\mathcal{O}}$ in codimension* 2.

2. Stratified Mukai flops

Consider the nilpotent orbit $\mathcal{O} = \mathcal{O}_{[2^k, 1^{n-2k}]}$ in $\mathfrak{sl}_n$, where $2k \leqslant n$. The closure $\overline{\mathcal{O}}$ admits exactly two symplectic resolutions given by

$$T^*G(k,n) \xrightarrow{\pi} \overline{\mathcal{O}} \xleftarrow{\pi^+} T^*G(n-k,n),$$

where $G(k,n)$ (resp. $G(n-k,n)$) is the Grassmannian of k (resp $n-k$) dimensional subspaces in $\mathbb{C}^n$. Let ϕ be the induced birational map $T^*G(k,n) \dashrightarrow T^*G(n-k,n)$.

It is shown by Namikawa ([8] Lemma 3.1) that when $2k < n$, π and π^+ are both small and the diagram is a flop. This is the *stratified Mukai flop of type $A_{k,n}$*. When $2k = n$, the birational map ϕ is an isomorphism.

Lemma 2.1. *If $n \neq 2k+1$, then ϕ is an isomorphism in codimension* 2. *If $n = 2k+1$, then ϕ is a MET over $\overline{\mathcal{O}}$ in codimension* 2.

Proof. The closure $\overline{\mathcal{O}}$ consists of orbits $\{\mathcal{O}_{[2^i, 1^{n-2i}]}\}_{0 \leqslant i \leqslant k}$. The fiber of π (resp. π^+) over a point in $\mathcal{O}_{[2^i, 1^{n-2i}]}$ is isomorphic to $G(k-i, n-2i)$ (resp. $G(n-k-i, n-2i)$). By a simple dimension count, one shows that the complement of $\pi^{-1}(\mathcal{O})$ (resp. $(\pi^+)^{-1}(\mathcal{O})$) is of codimension greater than 2 when $n \neq 2k+1$, which proves that ϕ is isomorphic in codimension 2.

Now suppose that $n = 2k+1$. Let Y be the nilpotent orbit $\mathcal{O}_{[2^{k-1}, 1^3]}$ and P (resp. P^+) the preimage of Y under π (resp. π^+). Then P is the subvariety

$$\{([F], x) \in G(k, 2k+1) \times Y \mid \mathrm{Im}(x) \subset F \subset \mathrm{Ker}(x)\}$$

in $T^*G(k, 2k+1) \subset G(k, 2k+1) \times \overline{\mathcal{O}}$. The induced map $P \to Y$ makes P a $\mathbb{P}^2$-bundle over Y. Similarly P^+ is the subvariety

$$\{([F^+], x) \in G(k+1, 2k+1) \times Y \mid \mathrm{Im}(x) \subset F^+ \subset \mathrm{Ker}(x)\}$$

in $T^*G(k+1, 2k+1)$. The map $P^+ \to Y$ makes P^+ a $\mathbb{P}^2$-bundle over Y.

Let $U = \mathcal{O} \cup Y$, which is open in $\overline{\mathcal{O}}$. The complement of $\pi^{-1}(U)$ (resp. $(\pi^+)^{-1}(U)$) is of codimension greater than 2. Notice that the $\mathbb{P}^2$-bundle P over Y is the dual of the $\mathbb{P}^2$-bundle P^+ over Y. One deduces that the diagram $\pi^{-1}(U) \to U \leftarrow (\pi^+)^{-1}(U)$ is a MET over U with center P, which concludes the proof. □

Notice that the precedent proof gives an explicit description of the center of the MET, which will be used later.

Now we introduce the stratified Mukai flops of type D. Let $\mathcal{O}$ be the orbit $\mathcal{O}_{[2^{k-1}, 1^2]}$ in $\mathfrak{so}_{2k}$, where $k \geqslant 3$ is an odd integer. Let $G^+_{\mathrm{iso}}(k)$, $G^-_{\mathrm{iso}}(k)$ be the two connected components of the orthogonal Grassmannian of k-dimensional isotropic subspaces in $\mathbb{C}^{2k}$ (endowed with a fixed non-degenerate symmetric form). Then we have two symplectic resolutions $T^*G^+_{\mathrm{iso}}(k) \to \overline{\mathcal{O}} \leftarrow T^*G^-_{\mathrm{iso}}(k)$. It is shown in [8] (Lemma 3.2) that this diagram is a flop and the two resolutions are both small.

Let ϕ be the induced birational map from $T^*G^+_{\mathrm{iso}}(k)$ to $T^*G^-_{\mathrm{iso}}(k)$. Then a simple dimension count shows that:

Lemma 2.2. *ϕ is an isomorphism in codimension* 2.

3. $\mathfrak{g} = \mathfrak{sl}_n$

Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{sl}_n$ corresponding to the partition $\mathbf{d} = [d_1, \ldots, d_k]$ and $x \in \mathcal{O}$. Let $(p_1, \ldots, p_s)$ be a sequence of integers such that $d_i = \sharp\{j \mid p_j \geqslant i\}$. Fix a flag $F := \{F_i\}$ of $\mathbb{C}^n$ of type $(p_1, \ldots, p_s)$ such that $xF_i \subset F_{i-1}$ for all i. Such a flag is called a *polarization* of x. Every nilpotent element has only finitely many different polarizations.

Assume that $p_{j-1} < p_j$ for some j. Consider the map $\alpha : F_j \to F_j/F_{j-2}$. The element x induces $\bar{x} \in \mathrm{End}(F_j/F_{j-2})$. We define a flag F' by $F'_i = F_i$ if $i \neq j-1$ and $F'_{j-1} = \alpha^{-1}(\mathrm{Ker}(\bar{x}))$. By Lemma 4.1 [8], F' is again a polarization of x with type $(p_1, \ldots, p_j, p_{j-1}, \ldots, p_s)$.

Let P (resp. P') be the stabilizer of F (resp. F') in $G = SL_n$. Then we obtain two symplectic resolutions $T^*(G/P) \xrightarrow{\pi} \overline{\mathcal{O}} \xleftarrow{\pi'} T^*(G/P')$. Let $\phi : T^*(G/P) \dashrightarrow T^*(G/P')$ be the induced birational map.

Lemma 3.1. (i) *If $p_j \neq p_{j-1} + 1$, then ϕ is isomorphic in codimension* 2;
(ii) *If $p_j = p_{j-1} + 1$, then ϕ is a MET over $\overline{\mathcal{O}}$ in codimension* 2.

Proof. To simplify the notations, set $r = p_{j-1}$ and $m = p_j + p_{j-1}$. Let $\overline{F}$ be the flag obtained from F by deleting the subspace F_{j-1}, which is also obtained from F' by the same manner. We denote by $\overline{P} \subset G$ the stabilizer of $\overline{F}$. Let X be the subvariety in $G/\overline{P} \times \overline{\mathcal{O}}$ consisting of the points $(\overline{E}, y)$ such that (i) $y\overline{E}_i \subset \overline{E}_{i-1}$ for $i \neq j-1$ and $y\overline{E}_{j-1} \subset \overline{E}_{j-1}$; (ii) the induced map $\bar{y} \in \mathrm{End}(\overline{E}_{j-1}/\overline{E}_{j-2})$ satisfies $\bar{y}^2 = 0$ and $\mathrm{rank}(\bar{y}) \leqslant r$.

The projection to the second factor of $G/\overline{P} \times \overline{\mathcal{O}}$ induces a morphism $pr : X \to \overline{\mathcal{O}}$. By the proof of Lemma 4.3 [8], the resolutions π, π' factorize through the map pr, which gives a diagram $T^*(G/P) \xrightarrow{\mu} X \xleftarrow{\mu'} T^*(G/P')$. By Lemma 4.3 [8], this diagram is locally a trivial family of stratified Mukai flops of type $A_{r,m}$. By Lemma 2.1, if $m \neq 2r+1$, then ϕ is isomorphic in codimension 2, which proves claim (i).

Now assume that $m = 2r+1$. Let Y be the subvariety in X consists of the points $(\overline{E}, y)$ such that the induced map $\bar{y} \in \mathrm{End}(\overline{E}_{j-1}/\overline{E}_{j-2})$ has rank $r-1$. By the proof of Lemma 2.1 and Lemma 4.3 [8], the diagram $T^*(G/P) \xrightarrow{\mu} X \xleftarrow{\mu'} T^*(G/P')$ is a MET over X in codimension 2 with center Y.

Let $\mathbf{d}'$ be the partition of n given by (possibly we need to re-order these parts):

$$d'_i = \begin{cases} d_i, & \text{if } i \neq r, r+2, \\ d_r - 1, & \text{if } i = r, \\ d_{r+2} + 1, & \text{if } i = r+2. \end{cases}$$

Then one can verify that the morphism $pr : X \to \overline{\mathcal{O}}$ maps Y isomorphically to the nilpotent orbit $\mathcal{O}_{\mathbf{d}'}$, which shows that the diagram $T^*(G/P) \xrightarrow{\pi} \overline{\mathcal{O}} \xleftarrow{\pi'} T^*(G/P')$ is a MET in codimension 2 over $\overline{\mathcal{O}}$ with center $\mathcal{O}_{\mathbf{d}'}$. $\square$

Notice that the precedent proof gives an explicit way to find out the MET center in $\overline{\mathcal{O}}$. Here we give an example.

Example 3.2. (Example 4.6 [8]) Let $\mathcal{O} = \mathcal{O}_{[3,2,1]} \subset \mathfrak{sl}_6$ and $x \in \mathcal{O}$. Then x has six polarizations $P_{\sigma(1),\sigma(2),\sigma(3)}$ of flag type $(\sigma(1), \sigma(2), \sigma(3))$, where σ are permutations. Let $Y_{i,j,k} = T^*(SL_6/P_{i,j,k})$, which gives a symplectic resolution for $\overline{\mathcal{O}}$. Then $Y_{321} \dashrightarrow Y_{231}$ is a MET in codimension 2 with center $\mathcal{O}_{[3,1^3]}$; $Y_{231} \dashrightarrow Y_{213}$ is isomorphic in codimension 2; $Y_{213} \dashrightarrow Y_{123}$ is a MET in codimension 2 with center $\mathcal{O}_{[2^3]}$ and so on. If a center appears twice in a sequence, then it is not really a MET center. For example, the birational map $Y_{321} \dashrightarrow Y_{132}$ is a MET in codimension 2 with center $\mathcal{O}_{[2^3]}$, but over the orbit $\mathcal{O}_{[3,1^3]}$, it is an isomorphism.

4. $\mathfrak{g} = \mathfrak{so}(V)$ or $\mathfrak{sp}(V)$

Let V be an n-dimensional vector space endowed with a non-degenerate bilinear symmetric (resp. anti-symmetric) form for $\mathfrak{g} = \mathfrak{so}(V)$ (resp. $\mathfrak{g} = \mathfrak{sp}(V)$). Let $\epsilon = 0$ if $\mathfrak{g} = \mathfrak{so}(V)$ and $\epsilon = 1$ if $\mathfrak{g} = \mathfrak{sp}(V)$.

Let $P_\epsilon(n)$ be the set of partitions $\mathbf{d}$ of n such that $\sharp\{i \mid d_i = l\}$ is even for every integer l with $l \equiv \epsilon \pmod 2$. These are exactly those partitions which appear as the Jordan types of nilpotent elements of $\mathfrak{so}(V)$ or of $\mathfrak{sp}(V)$. Let q be a non-negative integer such that $q \neq 2$ if $\epsilon = 0$. Define $\mathrm{Pai}(n, q)$ to be the set of partitions $\mathbf{e}$ of n such that $e_i \equiv 1 \pmod 2$ if $i \leqslant q$ and $e_i \equiv 0 \pmod 2$ if $i > q$. For $\mathbf{e} \in \mathrm{Pai}(n, q)$, let $I(\mathbf{e}) = \{j \mid j \equiv n+1 \pmod 2,\ e_j \equiv \epsilon \pmod 2,\ e_j \geqslant e_{j+1} + 2\}$.

The Spaltenstein map $S:\mathrm{Pai}(n,q)\to P_\epsilon(n)$ is defined as

$$S(\mathbf{e})_j=\begin{cases} e_j-1, & \text{if } j\in I(\mathbf{e}),\\ e_j+1, & \text{if } j-1\in I(\mathbf{e}),\\ e_j, & \text{otherwise.}\end{cases}$$

It is proved in [6] that for a nilpotent element of type $\mathbf{d}$, its polarization types are determined by $S^{-1}(\mathbf{d})$. For a sequence of integers $(p_1,\dots,p_k)$, we define $\mathbf{e}=\mathrm{ord}(p_1,\dots,p_k)$ to be the partition given by $e_i=\sharp\{j\mid p_j\geqslant i\}$.

Let $\mathcal{O}$ be a nilpotent orbit of type $\mathbf{d}$ in $\mathfrak{g}$ and $x\in\mathcal{O}$. Let $(p_1,\dots,p_k,q,p_k,\dots,p_1)$ be a sequence of integers such that $\mathbf{e}:=\mathrm{ord}(p_1,\dots,p_k,q,p_k,\dots,p_1)$ is in $\mathrm{Pai}(n,q)$ and $S(\mathbf{e})=\mathbf{d}$. Let F be an isotropic flag (i.e. $F_i^{\perp}=F_{2k+1-i},\forall i$) in V of type $(p_1,\dots,p_k,q,p_k,\dots,p_1)$ such that $xF_i\subset F_{i-1}$.

Assume that $p_{j-1}<p_j$ for some j. Consider the map $\alpha:F_j\to F_j/F_{j-2}$. The element x induces $\bar{x}\in\mathrm{End}(F_j/F_{j-2})$. We define another flag F' by $F'_i=F_i$ if $i\neq j-1,2k+2-j$, $F'_{j-1}=\alpha^{-1}(\mathrm{Ker}(\bar{x}))$ and $F'_{2k+2-j}=(F'_{j-1})^{\perp}$. By Lemma 4.2 [8], F' is again a polarization of x. We denote by P (resp. P') the stabilizer of F (resp. F'). Then we obtain two symplectic resolutions $T^*(G/P)\xrightarrow{\pi}\overline{\mathcal{O}}\xleftarrow{\pi'}T^*(G/P')$. Let ϕ be the induced birational map from $T^*(G/P)$ to $T^*(G/P')$.

Lemma 4.1. (i) *If $p_j\neq p_{j-1}+1$, then ϕ is isomorphic in codimension* 2;
(ii) *If $p_j=p_{j-1}+1$, then ϕ is a MET in codimension* 2 *over $\overline{\mathcal{O}}$.*

The proof goes along the same line as that in Lemma 3.1. The difference is the definition of the partition $\mathbf{d}'$ in the proof of (ii). Here we have $r=p_{j-1}$ and $p_j=r+1$. Let $\mathbf{e}'$ be the partition (after re-ordering if necessary) defined by

$$e'_j=\begin{cases} e_j, & \text{if } j\neq r,r+2,\\ e_r-2, & \text{if } j=r,\\ e_{r+2}+2, & \text{if } j=r+2.\end{cases}$$

Then $\mathbf{e}'\in\mathrm{Pai}(n,q)$. Now we should define $\mathbf{d}'=S(\mathbf{e}')$. In this case, ϕ is a MET in codimension 2 over $\overline{\mathcal{O}}$ with center $\mathcal{O}_{\mathbf{d}'}$.

Example 4.2. (Example 4.7 [8]) Let $\mathcal{O}=\mathcal{O}_{[4^2,1^2]}$ be the nilpotent orbit in $\mathfrak{so}_{10}$. Take an element $x\in\mathcal{O}$, then x has four polarizations P^+_{3223}, P^-_{3223}, P^+_{2332}, P^-_{2332}. Let $Y^+_{3223}=T^*(G/P^+_{3223})$ and so on. Then $Y^+_{3223}\dashrightarrow Y^+_{2332}$ is a MET in codimension 2 over $\overline{\mathcal{O}}$ with center $\mathcal{O}_{[3^2,2^2]}$.

5. Proof of Theorem 1.1

Let $\mathcal{O}$ be a nilpotent orbit in a classical simple Lie algebra $\mathfrak{g}$. By [4], every (proper) symplectic resolution for $\overline{\mathcal{O}}$ is of the form $T^*(G/P)\to\overline{\mathcal{O}}$ for some polarization P of $\mathcal{O}$. Assume that we have two symplectic resolutions $T^*(G/P_i)\to\overline{\mathcal{O}}$, $i=1,2$, then by the proof of Theorem 4.4 [8], we can reach $T^*(G/P_2)\to\overline{\mathcal{O}}$ from $T^*(G/P_1)\to\overline{\mathcal{O}}$ by using the operations in Section 2.2 and 2.3, possibly by using another operation which is a locally trivial family of stratified Mukai flops of type D (thus isomorphic in codimension 2 by Lemma 2.2). Now Lemmas 3.1 and 4.1 give the theorem.

References

[1] A. Beauville, Symplectic singularities, Invent. Math. 139 (2000) 541–549.
[2] D. Burns, Y. Hu, T. Luo, HyperKähler manifolds and birational transformations in dimension 4, in: Vector Bundles and Representation Theory, Columbia, MO, 2002, in: Contemp. Math., vol. 322, Amer. Math. Soc., Providence, RI, 2003, pp. 141–149.
[3] K. Cho, Y. Miyaoka, N.I. Shepherd-Barron, Characterizations of projective space and applications to complex symplectic manifolds, in: Higher Dimensional Birational Geometry, Kyoto, 1997, in: Adv. Stud. Pure Math., vol. 35, Math. Soc. Japan, Tokyo, 2002, pp. 1–88.
[4] B. Fu, Symplectic resolutions for nilpotent orbits, Invent. Math. 151 (2003) 167–186.
[5] B. Fu, Mukai flops and deformations of symplectic resolutions, math.AG/0510347, Math. Z., in press.
[6] W. Hesselink, Polarization in the classical groups, Math. Z. 160 (1978) 217–234.
[7] Y. Hu, S.-T. Yau, HyperKähler manifolds and birational transformations, Adv. Theor. Math. Phys. 6 (3) (2002) 557–574.
[8] Y. Namikawa, Birational geometry of symplectic resolutions of nilpotent orbits, math.AG/0404072.
[9] J. Wierzba, J. Wiśniewski, Small contractions of symplectic 4-folds, Duke Math. J. 120 (1) (2003) 65–95.