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Partial Differential Equations

Periodic unfolding and Robin problems in perforated domains

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Abstract

The periodic unfolding method was introduced in 2002 by D. Cioranescu et al. for the study of classical periodic homogenization. In this Note, we extend this method to perforated domains introducing also a boundary unfolding operator. As an application, we study the homogenization of some elliptic problems with Robin condition on the boundary of the holes. ***To cite this article: D. Cioranescu et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Éclatement périodique et homogénéisation du problème de Fourier. Dans cette Note nous adaptons la méthode d'éclatement périodique introduite par D. Cioranescu et al. en 2002 aux domaines perforés. Afin d'étudier des problèmes non homogènes, nous introduisons un opérateur d'éclatement frontière. Les résultats sont ensuite appliqués à l'homogénéisation de quelques problèmes elliptiques avec une condition du type Fourier sur le bord des trous. ***Pour citer cet article : D. Cioranescu et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Nous adaptons ici au cas des domaines perforés la méthode d'éclatement périodique, une technique d'homogénéisation, introduite dans [4] par Cioranescu, Damlamian et Griso. L'intérêt de cette approche vient du fait qu'elle ne fait intervenir que les notions classiques de convergence dans des espaces du type L^p, ce qui rend les démonstrations des résultats d'homogénéisation assez simples. De plus, elle permet d'avoir des estimations d'erreurs et des résultats de correcteurs (voir [4] et [8]). La méthode est basée sur deux ingrédients : un opérateur d'éclatement périodique et une décomposition permettant de séparer les échelles micro et macroscopiques des fonctions.

Le cadre géométrique de notre travail est le suivant : soit Ω un domaine borné de $\mathbb{R}^N$ tel que $|\partial\Omega| = 0$, $Y = \prod_{i=1}^{N}[0, \ell_i[^N$ la cellule de référence, T un ouvert inclus dans Y tel que ∂T ne contient pas les sommets de Y et $Y^\star = Y \setminus \overline{T}$. Le domaine perforé Ω^ε (supposé connexe) est défini par $\Omega^\varepsilon = \Omega \setminus T^\varepsilon$, où $T^\varepsilon = \{\bigcup \varepsilon(\xi + T),\ \xi \in \mathbb{Z}^N\}$. Alors, l'opérateur d'éclatement périodique $\mathcal{T}_\varepsilon$ sur le domaine perforé Ω^ε est défini par (1). On définit de même

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dans (2), un opérateur d'éclatement $\mathcal{T}_\varepsilon^b$ sur le bord ∂T^ε. Les propriétés de ces opérateurs pour des fonctions dans des espaces L^p sont données dans les Propositions 2 et 4.

La décomposition macro-micro pour des fonctions définies sur des domaines perforés, est introduite au paragraphe 3. Comme dans le cas des domaines fixes (voir [4]), elle est inspirée par la méthode des éléments finis. Grâce à cette décomposition, on peut enoncer le Théorème 6, caractérisant la convergence de suites bornées φ_ε dans l'espace $H^1(\Omega^\varepsilon)$.

Par rapport au cas des domaines fixes, les convergences de ce théorème ont lieu seulement localement, conséquence du fait que les trous peuvent rencontrer la frontière $\partial\Omega$, situation qui était interdite dans les papiers précédents traitant les domaines perforés (voir [5,6] et [7]). De plus, il faut noter que les opérateurs d'éclatement (1) et (2) transforment des fonctions définies sur les domains oscillants Ω^ε, resp. ∂T^ε, en des fonctions définies sur les domains fixes $\Omega \times Y^\star$, resp. $\Omega \times \partial T$ (c'est d'ailleurs l'un de leurs principaux intérêts). Ceci permet de parler de la convergence des éclatés $\mathcal{T}_\varepsilon\varphi_\varepsilon$ dans l'espace (fixe) $L^2_{\text{loc}}(\Omega; H^1(Y^\star))$, sans être obligé d'introduire des opérateurs de prolongement de φ_ε comme dans les articles cités ci-dessus. Ceci implique que l'on peut traiter des situations plus générales que celles de ces travaux. En particulier, pour le cas du problème de Neumann homogène, on peut considérer certains trous fractales comme le flocon de neige bidimensionnel considéré dans [9]. Le résultat d'homogénéisation (Théorème 7) pour le cas d'une condition de Robin générale (étudié dans [5]), est prouvé ici sous la seule hypothèse que la frontière $\partial\Omega^\varepsilon$ est de Lipschitz. Dans la preuve de ce théorème, un rôle essentiel est joué par la Proposition 5, dont la démonstration repose sur les propriétés de l'opérateur d'éclatement frontière.

1. Introduction

The periodic unfolding method was introduced in [4] by Cioranescu, Damlamian and Griso for the study of periodic or multi-scale homogenization in the case of fixed domains. This method is based on two ingredients: the unfolding operator and a macro-micro decomposition of functions, allowing to separate the macroscopic and microscopic scales. The interest of the method comes from the fact that it only deals with functions in classical L^p spaces and related convergences. This renders the proof of homogenization results quite elementary. It also provides error estimates and corrector results (see [8] for the case of fixed domains).

We adapt here the method to domains with holes, defining an unfolding operator for functions defined on periodically perforated domains. We apply it to elliptic problems with nonhomogeneous boundary conditions on the holes (Neumann or Robin type). To do so, we are led to define a boundary unfolding operator and to study its properties. The main feature of the method based on unfolding operators, is that they map functions defined on oscillating domains (like perforated ones or the boundaries of the holes), into functions defined on fixed domains, so that no extension operators are needed. This allows us to treat a larger class of geometrical situations than, for instance, in [5,6], and [7]. In particular, for the homogeneous Neumann problem, we can consider some fractal holes like the two-dimensional snowflake (see [9]). For a general nonhomogeneous Robin condition, we only assume a Lipschitz boundary (which is sufficient to give a sense to traces in Sobolev spaces).

2. Unfolding operators in perforated domains and boundary unfolding operators

Let Ω be an open bounded set in $\mathbb{R}^N$, $Y = \prod_{i=1}^N [0, \ell_i[^N$, T an open set included in Y such that ∂T does not contain the summits of Y and $Y^\star = Y \setminus \overline{T}$. The perforated domain Ω^ε is defined by

$$\Omega^\varepsilon = \Omega \setminus T^\varepsilon, \quad \text{where } T^\varepsilon = \bigcup_{\xi \in \mathbb{Z}^N} \varepsilon(\xi + T).$$

We assume in the following that Ω^ε is a connected set. Unlike preceding papers treating perforated domains (see for example [5,6], and [7]) we can allow that the holes meet the boundary $\partial\Omega$. In the sequel we suppose that the domain Ω is such that

$$|\partial\Omega| = 0,$$

which is equivalent to the fact that the number of cells εY intersecting the boundary $\partial\Omega$ is of the order of ε^{-N}.

For $z \in \mathbb{R}^N$, $[z]_Y$ denotes the unique integer combination $\sum_{j=1}^N k_j \ell_j$, such that $z - [z]_Y$ belongs to Y. Set $\{z\}_Y = z - [z]_Y$. Then for almost every $x \in \mathbb{R}^N$, there exists a unique element in $\mathbb{R}^N$ denoted by $[x/\varepsilon]_Y$ such that

$$x - \varepsilon\left[\frac{x}{\varepsilon}\right]_Y = \varepsilon\left\{\frac{x}{\varepsilon}\right\}_Y, \quad \text{where } \left\{\frac{x}{\varepsilon}\right\}_Y \in Y.$$

In the sequel, we will use the following notation:

- $\tilde{\varphi}$ for the extension by zero outside Ω^ε (resp. Ω) for any function φ in $L^p(\Omega^\varepsilon)$ (resp. $L^p(\Omega)$),
- θ for the proportion of the material in the elementary cell, i.e. $\theta = |Y^\star|/|Y|$,
- $\rho(Y)$ for the diameter of the cell Y.

Definition 1 *(the unfolding operator $\mathcal{T}_\varepsilon$ in Ω^ε).* Let $\varphi \in L^p(\Omega^\varepsilon)$, $p \in [1, +\infty]$. We define the function $\mathcal{T}_\varepsilon(\varphi) \in L^p(\mathbb{R}^N \times Y^\star)$ by

$$\mathcal{T}_\varepsilon(\varphi)(x, y) = \tilde{\varphi}\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) \quad \text{for } (x, y) \in \mathbb{R}^N \times Y^\star. \tag{1}$$

The main properties of the unfolding operator $\mathcal{T}_\varepsilon$, given in [4] for nonperforated (fixed) domains, can easily be adapted for perforated ones. We list here only those that will be used in the sequel.

Proposition 2. *The unfolding operator $\mathcal{T}_\varepsilon$ has the following properties*:

(i) *$\mathcal{T}_\varepsilon(\varphi\psi) = \mathcal{T}_\varepsilon(\varphi)\mathcal{T}_\varepsilon(\psi)$, for all φ and ψ in $L^p(\Omega^\varepsilon)$.*
(ii) *Let φ in $L^p(Y)$ or in $L^p(Y^\star)$ and set $\varphi^\varepsilon(x) = \varphi(x/\varepsilon)$. Then $\mathcal{T}_\varepsilon(\varphi^\varepsilon)(x, y) = \varphi(y)$.*
(iii) *Let $\varphi \in L^1(\Omega^\varepsilon)$. Then one has the following integration formula*:

$$\int_{\Omega^\varepsilon} \varphi \,\mathrm{d}x = \frac{1}{|Y|}\int_{\mathbb{R}^N \times Y^\star} \mathcal{T}_\varepsilon(\varphi)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{|Y|}\int_{\widetilde{\Omega}^\varepsilon \times Y^\star} \mathcal{T}_\varepsilon(\varphi)\,\mathrm{d}x\,\mathrm{d}y,$$

where the set $\widetilde{\Omega}^\varepsilon$ is the smallest finite union of εY cells containing Ω.
(iv) *Let $\varphi \in L^2(\Omega)$. Then $\mathcal{T}_\varepsilon(\varphi) \to \tilde{\varphi} \otimes 1$ strongly in $L^2(\mathbb{R}^N \times Y^\star)$.*
(v) *Let $\{\varphi^\varepsilon\}$ be a bounded sequence in $L^2(\Omega^\varepsilon)$ with $\mathcal{T}_\varepsilon(\varphi^\varepsilon) \rightharpoonup \hat{\varphi}$ weakly in $L^2(\mathbb{R}^N \times Y^\star)$. Then*

$$\widetilde{\varphi^\varepsilon} \rightharpoonup \frac{1}{|Y|}\int_{Y^\star} \hat{\varphi}(\cdot, y)\,\mathrm{d}y \quad \textit{weakly in } L^2(\mathbb{R}^N).$$

Definition 3 *(the boundary unfolding operator $\mathcal{T}_\varepsilon^b$ on $\partial\Omega^\varepsilon$).* Let $\varphi \in L^p(\partial T^\varepsilon)$, $p \in [1, +\infty]$. We define the function $\mathcal{T}_\varepsilon^b(\varphi) \in L^2(\mathbb{R}^N \times \partial T)$ by

$$\mathcal{T}_\varepsilon^b(\varphi)(x, y) = \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) \quad \text{for } (x, y) \in \mathbb{R}^N \times \partial T. \tag{2}$$

The operator $\mathcal{T}_\varepsilon^b$ has properties similar to those given in Proposition 2, in particular (i) and (ii) are obvious. We have an integration formula for functions defined on the boundary ∂T^ε as well as the corresponding formulation for (iv) and (v). They are recalled in the next result whose proof follows from definition (3), by using the change of variables herein.

Proposition 4.

(i) *For every $\varphi \in L^2(\partial T^\varepsilon)$, one has*

$$\int_{\partial T^\varepsilon} \varphi(x)\,\mathrm{d}\sigma_x = \frac{1}{\varepsilon|Y|}\int_{\mathbb{R}^N \times \partial T} \mathcal{T}_\varepsilon^b\varphi(x, y)\,\mathrm{d}x\,\mathrm{d}\sigma_y = \frac{1}{\varepsilon|Y|}\int_{\widetilde{\Omega}^\varepsilon \times \partial T} \mathcal{T}_\varepsilon^b\varphi(x, y)\,\mathrm{d}x\,\mathrm{d}\sigma_y.$$

(ii) *Let* $\varphi \in H^1(\Omega)$. *Then* $\mathcal{T}_\varepsilon^b(\varphi) \to \tilde{\varphi} \otimes 1$ *strongly in* $L^2(\mathbb{R}^N \times \partial T)$.
(iii) *Let* $\{\varphi^\varepsilon\}$ *be a bounded sequence in* $L^2(\partial T^\varepsilon)$ *with* $\mathcal{T}_\varepsilon^b(\varphi^\varepsilon) \rightharpoonup \hat{\varphi}$ *weakly in* $L^2(\mathbb{R}^N \times \partial T)$. *Then*

$$\varepsilon \int_{\partial T^\varepsilon} \varphi^\varepsilon(x)\psi(x)\,\mathrm{d}\sigma_x \to \frac{1}{|Y|} \int_{\mathbb{R}^N \times \partial T} \hat{\varphi}(x,y)\psi(x)\,\mathrm{d}\sigma_y, \quad \forall \psi \in H^1(\Omega).$$

In the sequel (for the applications to Robin problems), the next result will play an essential role.

Proposition 5. *Let* $g \in L^2(\partial T)$ *and set* $g^\varepsilon(x) = g(x/\varepsilon)$ *for all* $x \in \mathbb{R}^N \setminus \bigcup \varepsilon(\xi + T)$ *where the union is taken for all* $\xi \in \mathbb{Z}^N$. *Let* φ *in* $H^1(\Omega)$. *Then one has the convergence*

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\partial T^\varepsilon} g^\varepsilon(x)\varphi(x)\,\mathrm{d}\sigma_x = \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g) \int_\Omega \varphi\,\mathrm{d}x, \tag{3}$$

where $\mathcal{M}_{\partial T}(g) = \frac{1}{|\partial T|} \int_{\partial T} g(y)\,\mathrm{d}\sigma_y$. *Furthermore, one has the estimate*

$$\left| \int_{\partial T^\varepsilon} g^\varepsilon(x)\varphi(x)\,\mathrm{d}\sigma_x \right| \leqslant \frac{C}{\varepsilon}\big(\left|\mathcal{M}_{\partial T}(g)\right| + \varepsilon\big) \|\nabla\varphi\|_{[L^2(\Omega^\varepsilon)]^N}. \tag{4}$$

Finally, if $\mathcal{M}_{\partial T}(g) = 0$, *then*

$$\lim_{\varepsilon \to 0} \int_{\partial T^\varepsilon} g^\varepsilon(x)\varphi(x)\,\mathrm{d}\sigma_x = 0. \tag{5}$$

Sketch of the proof. Due to density properties, it is enough to prove these assertions for all $\varphi \in D(\mathbb{R}^N)$. Assertion (3) follows simply from Proposition 4(i)–(ii). Observe now that from Definition 3 and Proposition 4(i), we have

$$\int_{\partial T^\varepsilon} g^\varepsilon(x)\varphi(x)\,\mathrm{d}\sigma_x = \frac{1}{\varepsilon|Y|} \int_{\mathbb{R}^N \times \partial T} g(y)\mathcal{T}_\varepsilon^b\varphi(x,y)\,\mathrm{d}x\,\mathrm{d}\sigma_y = \frac{1}{\varepsilon|Y|} \int_{\mathbb{R}^N} \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y\right)\left[\int_{\partial T} g(y)\,\mathrm{d}\sigma_y\right]\mathrm{d}x$$
$$+ \frac{1}{\varepsilon|Y|} \int_{\mathbb{R}^N \times \partial T} g(y)\left\{\varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) - \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y\right)\right\}\mathrm{d}x\,\mathrm{d}\sigma_y,$$

and (4) is then a consequence of the properties of φ. Taking $\varepsilon \to 0$ in the former equality gives immediately, since $\varphi \in D(\mathbb{R}^N)$, that

$$\int_{\partial T^\varepsilon} g^\varepsilon(x)\varphi(x)\,\mathrm{d}\sigma_x \to \frac{1}{|Y|} \int_{\partial T} y g(y)\,\mathrm{d}\sigma_y \int_{\mathbb{R}^N} \nabla\varphi(x)\,\mathrm{d}x = 0. \qquad \square$$

3. Convergence for sequences in $W^{1,p}(\Omega^\varepsilon)$

Following [4], we decompose any function φ in the form $\varphi = Q_\varepsilon(\varphi) + R_\varepsilon(\varphi)$, where R_ε is designed in order to capture the oscillations. As in the case of nonperforated domains, we start by defining $Q_\varepsilon(\varphi)$ on the nodes $\varepsilon\xi_k$ of the εY-lattice, by taking the average of φ on a small ball B_ε centered on $\varepsilon\xi_k$ and not touching the holes. However, B_ε must be entirely contained in Ω^ε. To guarantee that, we are led to define $Q_\varepsilon(\varphi)$ on a subdomain of Ω^ε only. To do so, for every $\delta > 0$, let us introduce the following domains: $\Omega_\delta^\varepsilon = \{x \in \Omega;\ d(x, \partial\Omega) > \delta\}$, and $\widehat{\Omega}_\delta^\varepsilon = \mathrm{int}(\bigcup_{\xi \in \Pi_\varepsilon^\delta} \varepsilon(\xi + \overline{Y}))$, where $\Pi_\varepsilon^\delta = \{\xi \in \mathbb{Z}^N;\ \varepsilon(\xi + \overline{Y}) \subset \Omega_\delta^\varepsilon\}$. Then, for every node $\varepsilon\xi_k$ in $\widehat{\Omega}_{2\varepsilon\rho(Y)}^\varepsilon$, we define

$$Q_\varepsilon(\varphi)(\varepsilon\xi_k) = \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon} \varphi(\varepsilon\xi_k + \varepsilon z)\,\mathrm{d}z.$$

The next step consist of taking a Q_1-interpolate of the discrete function $Q_\varepsilon(\varphi)(\varepsilon\xi_k)$, in order to get $Q_\varepsilon(\varphi)$ on all $\widehat{\Omega}_{2\varepsilon\rho(Y)}^\varepsilon$. Finally, on $\widehat{\Omega}_{2\varepsilon\rho(Y)}^\varepsilon$, R_ε will be defined as the remainder: $R_\varepsilon(\varphi) = \varphi - Q_\varepsilon(\varphi)$.

With this decomposition we can now state the main result of this section.

Theorem 6. *Let φ_ε be a bounded sequence in $H^1(\Omega^\varepsilon)$. Then, up to a subsequence, there exists φ in $H^1(\Omega)$ and $\hat{\varphi}$ in $L^2(\Omega; H^1_{\text{per}}(Y^\star))$ such that*

$$\begin{aligned}
&\mathcal{Q}_\varepsilon(\varphi_\varepsilon) \rightharpoonup \varphi \quad \text{weakly in } H^1_{\text{loc}}(\Omega),\\
&\mathcal{T}_\varepsilon(\varphi_\varepsilon) \rightharpoonup \varphi \quad \text{weakly in } L^2_{\text{loc}}\big(\Omega; H^1\big(Y^\star\big)\big),\\
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\mathcal{R}_\varepsilon(\varphi_\varepsilon)\big) \rightharpoonup \hat{\varphi} \quad \text{weakly in } L^2_{\text{loc}}\big(\Omega; H^1\big(Y^\star\big)\big),\\
&\mathcal{T}_\varepsilon\big(\nabla_x(\varphi_\varepsilon)\big) \rightharpoonup \nabla_x\varphi + \nabla_y\hat{\varphi} \quad \text{weakly in } L^2_{\text{loc}}\big(\Omega \times Y^\star\big).
\end{aligned}$$

When comparing with the case of fixed domains, the main difference is that, since the decomposition was done on $\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)} \subset \Omega$, we have here local convergences only.

Sketch of the proof. Let K be a compact set in Ω. As $d(K, \partial\Omega) > 0$, there exists $\varepsilon_K > 0$ depending on K, such that $\forall \varepsilon \leqslant \varepsilon_K$, one has $K \subset \widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}$. Hence,

$$\big\|\mathcal{Q}_\varepsilon(\varphi^\varepsilon)\big\|_{L^2(K)} \leqslant \big\|\mathcal{Q}_\varepsilon(\varphi^\varepsilon)\big\|_{L^2(\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)})} \leqslant C\|\varphi^\varepsilon\|_{H^1(\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)})} \leqslant C\|\varphi^\varepsilon\|_{H^1(\Omega)} \leqslant C,$$

so that there exists some $\varphi \in H^1(\Omega)$ such that $\mathcal{Q}_\varepsilon(\varphi^\varepsilon) \rightharpoonup \varphi$ weakly in $L^2_{\text{loc}}(\Omega)$. Also $\nabla \mathcal{Q}_\varepsilon(\varphi^\varepsilon)$ converges to some φ_1 weakly in $(L^2_{\text{loc}}(\Omega))^N$.

Let $\psi \in D(\Omega)$. There exists ε_ψ depending on $\operatorname{supp}\psi$ such that $\forall \varepsilon \leqslant \varepsilon_\psi$: $\operatorname{supp}\psi \subset \widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}$. Therefore,

$$\int_\Omega \nabla \mathcal{Q}_\varepsilon(\varphi^\varepsilon)\psi\,\mathrm{d}x = \int_{\widehat{\Omega}^\varepsilon_{2\varepsilon\rho(Y)}} \nabla \mathcal{Q}_\varepsilon(\varphi^\varepsilon)\psi\,\mathrm{d}x = -\int_\Omega \mathcal{Q}_\varepsilon(\varphi^\varepsilon)\operatorname{div}\psi\,\mathrm{d}x.$$

Passing to the limit as $\varepsilon \to 0$, yields the first convergence in Theorem 6. The fact that φ belongs actually to $H^1(\Omega)$ is then proved by using the Dominated Convergence theorem. □

4. Application: homogenization of a Robin problem

Now, we apply the periodic unfolding method to an elliptic problem with Fourier boundary conditions in a perforated domain (other cases will be treated in a forecoming paper). We start by defining some spaces that we shall use later on. We introduce

$$\begin{aligned}
&V_\varepsilon = \big\{\varphi \in H^1\big(\Omega^\varepsilon\big) \mid \varphi = 0 \text{ on } \partial\Omega\big\}, \qquad H^1_{\text{per},T_0}\big(Y^\star\big) = \big\{\psi \in H^1_{\text{per}}\big(Y^\star\big) \mid \psi = 0 \text{ in } T\big\},\\
&W_{\text{per}}\big(Y^\star\big) = \big\{v \in H^1_{\text{per}}\big(Y^\star\big) \mid \mathcal{M}_{Y^\star}(v) = 0\big\}, \quad \mathcal{W}_{\text{per}}\big(Y^\star\big) = H^1_{\text{per}}\big(Y^\star\big)/\mathbb{R}.
\end{aligned}$$

Let us consider the problem

$$\begin{cases}
-\operatorname{div}\big(A^\varepsilon\nabla u^\varepsilon\big) = f \quad \text{in } \Omega^\varepsilon, \quad u^\varepsilon = 0 \quad \text{on } \partial\Omega^\varepsilon,\\
\frac{\partial u^\varepsilon}{\partial n} + h\varepsilon u^\varepsilon = \varepsilon g^\varepsilon \qquad \text{on } \partial T^\varepsilon,
\end{cases} \tag{6}$$

where h is a real positive number, $f \in L^2(\Omega)$, $g^\varepsilon(x) = g(x/\varepsilon)$ where g is a Y-periodic function in $L^2(\partial T)$. The matrix $A^\varepsilon = A^\varepsilon(x) = (a^\varepsilon_{ij}(x))_{1\leqslant i,j\leqslant N}$ is measurable, bounded in $L^\infty(\Omega)$ and satisfies the uniform ellipticity condition $c|\xi|^2 \leqslant A^\varepsilon(x)\xi\cdot\xi \leqslant C|\xi|^2$ a.e. $x \in \Omega$, with strictly positive constants c and C.

The hypotheses that we are going to make now are weaker than the ones we would normally make when using other homogenization methods [1,2]. Let us suppose that

(H_1) If $h = 0$ and $g = 0$, we have the uniform (with respect to ε) Poincaré inequality in Ω^ε.
(H_2) If $h \neq 0$ or $g \neq 0$, T has a Lipschitz boundary.

Observe that (H_2) guarantees the existence of a uniform extension operator (see [3]) so a uniform Poincaré inequality in Ω^ε. Let us point out that under hypothesis (H_1) (the Neumann homogeneous case), we can treat the case of some fractal holes like the two dimensional snowflake (cf. [9]).

Theorem 7. *Suppose that f, g and A^ε satisfy the above hypotheses. Assume furthermore that $\mathcal{T}_\varepsilon(A^\varepsilon) \to A$ a.e. in $\Omega \times Y^\star$. Then there exists $u^0 \in H^1_0(\Omega)$ and $\hat{u} \in L^2(\Omega; H^1_{\mathrm{per}}(Y^\star))$ such that*

(i) $\widetilde{u^\varepsilon} \rightharpoonup \theta u^0$ *weakly in* $L^2(\Omega)$; $\mathcal{T}_\varepsilon(u^\varepsilon) \rightharpoonup u^0$ *weakly in* $L^2_{\mathrm{loc}}(\Omega; H^1(Y^\star))$,
(ii) $\frac{1}{\varepsilon}\mathcal{T}_\varepsilon(R_\varepsilon u^\varepsilon) \rightharpoonup \hat{u}$ *weakly in* $L^2_{\mathrm{loc}}(\Omega; H^1(Y^\star))$,
(iii) $\mathcal{T}_\varepsilon(\nabla u^\varepsilon) \rightharpoonup \nabla_x u^0 + \nabla_y \hat{u}$ *weakly in* $L^2_{\mathrm{loc}}(\Omega \times Y^\star)$.

The pair $(u_0, \hat{u})$ is the unique solution of the problem

$$\frac{1}{|Y^\star|}\int_{\Omega\times Y^\star} A(x,y)\big(\nabla_x u^0 + \nabla_y \hat{u}\big)\big(\nabla_x \varphi(x) + \nabla_y \psi(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y + h\frac{|\partial T|}{|Y^\star|}\int_\Omega u^0 \varphi\,\mathrm{d}x$$
$$= \int_\Omega f\varphi\,\mathrm{d}x + \frac{1}{|Y^\star|}\int_\Omega \varphi\,\mathrm{d}x \int_{\partial T} g\,\mathrm{d}\sigma_y, \quad \forall \varphi \in H^1_0(\Omega),\ \forall \psi \in L^2\big(\Omega; H^1_{\mathrm{per},T_0}\big(Y^\star\big)\big). \tag{7}$$

Sketch of the proof. Estimate (4) shows that

$$\|\nabla u^\varepsilon\|^2_{(L^2(\Omega^\varepsilon))^N} + h\varepsilon\|u^\varepsilon\|^2_{L^2(\partial T^\varepsilon)} \leqslant C\big(1+\varepsilon+\big|\mathcal{M}_{\partial T}(g)\big|\big)\|\nabla u^\varepsilon\|_{(L^2(\Omega^\varepsilon))^N},$$

which, by using the uniform Poincaré inequality and Theorem 6, gives convergences (i)–(iii). Eq. (7) is obtained by taking successively $\varphi \in D(\Omega)$, and $\varepsilon\varphi(\cdot)\xi(\frac{\cdot}{\varepsilon})$, with $\xi \in H^1_{\mathrm{per},T_0}(Y^\star)$, as test functions in (6), and passing to the limit thanks to Propositions 2, 4 and 5. □

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