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Partial Differential Equations

Bound and ground states of coupled nonlinear Schrödinger equations

Antonio Ambrosetti [1], Eduardo Colorado [2]

SISSA, 2-4, via Beirut, Trieste 34014, Italy

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Abstract

We prove existence of bound and ground states of some systems of coupled nonlinear Schrödinger equations. ***To cite this article: A. Ambrosetti, E. Colorado, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Solutions à énergie finie et minimale d'équations de Schrödinger non linéaires couplées. On montre l'existence de solutions à énergie finie et énergie minimale pour quelques systèmes couplés d'équations de Schrödinger non linéaires. ***Pour citer cet article : A. Ambrosetti, E. Colorado, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

On s'intéresse à l'existence des solutions de quelques systèmes couplés d'équations de Schrödinger non linéaires. Concernant le système (2) on montre qu'ils existent deux constantes $\Lambda' > \Lambda > 0$ qui dépendent uniquement de λ_1, λ_2, μ_1 et μ_2, telles que le système (2) admet une solution à énergie finie, (u, v), avec $u \not\equiv 0$, $v \not\equiv 0$, à condition que, soit $\beta < \Lambda$, ou $\beta > \Lambda'$. Si $\beta > 0$ on a $u > 0$ et $v > 0$. De plus, on montre que dans le cas où $\beta > \Lambda'$, la solution (u, v) est une solution à énergie minimale. Une comparaison avec un résultat récent [5] est faite dans les Remarques 3 et 5. Concernant le système (6) on montre l'existence de solutions de type « multi-bumps ».

1. Introduction

In spite of the interest that system of coupled NLS (Nonlinear Schrödinger) equations have in nonlinear Optics, see e.g. [1], only few rigorous general results have been proved so far. Here we will focus on systems of two coupled NLS equations like

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta g_u(u, v), \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta g_v(u, v), \end{cases} \tag{1}$$

E-mail addresses: ambr@sissa.it (A. Ambrosetti), colorado@sissa.it (E. Colorado).
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where $\lambda_i, \mu_i > 0$, $i = 1, 2$, β is a real parameter and $x \in \mathbb{R}^n$, $n = 2, 3$.[3]

We use the following notation: E denotes the Sobolev space $W^{1,2}(\mathbb{R}^n)$, with scalar product $(u \mid w) = \int (\nabla u \cdot \nabla v + uw)\,\mathrm{d}x$ and norm $\|u\|^2 = (u \mid u)$ (hereafter $\int$ means $\int_{\mathbb{R}^n}$); H denotes the space of radial $W^{1,2}(\mathbb{R}^n)$ functions; $(u \mid w)_i = \int (\nabla u \cdot \nabla w + \lambda_i uw)\,\mathrm{d}x$ and $\|u\|_i^2 = (u \mid u)_i$, $i = 1, 2$; and

$$\Phi(u, v) = I_1(u) + I_2(v) - \beta \int g(u, v)\,\mathrm{d}x, \quad \text{where } I_i(u) = \frac{1}{2}\|u\|_i^2 - \frac{1}{4}\mu_i \int u^4\,\mathrm{d}x,\ u \in E,\ i = 1, 2.$$

We say that $(u^*, v^*) \neq (0, 0)$ is a bound state of (1) if $\Phi'(u^*, v^*) = 0$. A solution $(\tilde{u}, \tilde{v})$ of (1) is called a ground state if $\tilde{u} > 0$, $\tilde{v} > 0$ and $\Phi(\tilde{u}, \tilde{v}) = \min\{\Phi(u, v)\colon (u, v) \in E \times E \setminus \{(0, 0)\}, \Phi'(u, v) = 0\}$.

More details and further results are contained in [3].

2. Global results

Motivated by the recent paper [5], we deal here with the case in which $g(u, v) = \frac{1}{2}u^2v^2$, namely with the system

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta u v^2, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v. \end{cases} \tag{2}$$

To state our existence results, we need to introduce some notation. Let U be denote the unique positive radial solution of $-\Delta u + u = u^3$ and set

$$U_i(x) = \sqrt{\frac{\lambda_i}{\mu_i}}\, U\big(\sqrt{\lambda_i}\, x\big), \quad i = 1, 2.$$

Remark that these U_i are solutions of $-\Delta u + \lambda_i u = \mu_i u^3$. We also denote by γ_i, $i = 1, 2$, the following positive constants which depend on λ_i, μ_i, $i = 1, 2$, by setting

$$\gamma_1^2 = \inf_{\varphi \in H \setminus \{0\}} \frac{\|\varphi\|_2^2}{\int U_1^2 \varphi^2}, \qquad \gamma_2^2 = \inf_{\varphi \in H \setminus \{0\}} \frac{\|\varphi\|_1^2}{\int U_2^2 \varphi^2}.$$

It is worth pointing out immediately that later on we will show that

$$\mu_j \left(\frac{\lambda_i}{\lambda_j}\right)^{1-n/4} \leqslant \gamma_j^2 \leqslant \max\left\{\mu_j \left(\frac{\lambda_i}{\lambda_j}\right)^{1-n/2}, \mu_j \frac{\lambda_i}{\lambda_j}\right\}, \quad i, j = 1, 2,\ i \neq j. \tag{3}$$

Our first result is concerned with existence of bound states.

Theorem 1. *Let $\Lambda = \min\{\gamma_1^2, \gamma_2^2\}$. Then* (2) *has a radial bound state (u^*, v^*), provided that $\beta < \Lambda$. If $\beta > 0$ these bound states are positive, namely $u^* > 0$, $v^* > 0$.*

Our second result deals with ground states.

Theorem 2. *Let $\Lambda' = \max\{\gamma_1^2, \gamma_2^2\}$. Then* (2) *has a radial ground state $(\tilde{u}, \tilde{v})$ provided $\beta > \Lambda'$.*

Remark 3. Theorem 1 is related to [5, Theorem 2]. Theorem 2 is new.[4] In particular, there are no existence result in [5] dealing with $\beta \gg 1$. For further comparisons with [5], see Remark 5.

In the rest of the section, we will give an outline of the proofs of Theorems 1 and 2. We will work in $H \times H$. We set $\gamma(u, v) = (I_1'(u)|u) + (I_2'(v)|v) - 2\beta \int u^2 v^2\,\mathrm{d}x$ and

$$\mathcal{M} = \big\{(u, v) \in H \times H \setminus \big\{(0, 0)\big\}\colon \gamma(u, v) = 0\big\}.$$

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[3] We could also consider a system in $\mathbb{R}^n$, with a nonlinearity of the type u^p, provided $p < \frac{n+2}{n-2}$.

[4] After this paper was completed, we learned that some similar results have been obtained independently in the preprint "Positive solutions for a weakly coupled nonlinear Schrödinger system" by L. Maia, E. Montefusco and B. Pellacci. We thank Simone Secchi who brought to our attention this paper.

One has that

$$\big(\nabla\gamma(u,v)\mid(u,v)\big) = -2\big(\|u\|_1^2+\|v\|_2^2\big) < 0, \quad \forall (u,v)\in\mathcal{M}, \tag{4}$$

and thus $\mathcal{M}$ is a smooth manifold locally near any point $(u,v)\neq(0,0)$, such that $\gamma(u,v)=0$. Moreover, from $\Phi''(0,0)[(\phi_1,\phi_2)]^2 = I_1''(0)[\phi_1]^2 + I_2''(0)[\phi_2]^2 > 0$, $\forall(\phi_1,\phi_2)\in E\times E\setminus\{(0,0)$, we infer that $(0,0)$ is a strict minimum for Φ. As a consequence, $(0,0)$ is an isolated point of the set $\{\gamma(u,v)=0\}$, proving that $\mathcal{M}$ is a smooth complete manifold. Furthermore, (4) also implies that any critical point of Φ constrained on $\mathcal{M}$ is a critical point of Φ. Note that for $(u,v)\in\mathcal{M}$ one has that $\Phi(u,v)=\frac14(\|u\|_1^2+\|v\|_2^2)>0$.

Since U_i satisfy $-\Delta u+\lambda_i u=\mu_i u^3$, then $(U_1,0)$ and $(0,U_2)$ are particular solutions of (2). We also put

$$\mathcal{N}_i = \big\{u\in H\colon \big(I_i'(u)|u\big)_i = 0\big\} = \left\{u\in H\colon \|u\|_i^2-\mu_i\int u^4=0\right\}, \quad i=1,2.$$

One has that $U_i\in\mathcal{N}_i$ and $\phi\in T_{U_i}\mathcal{N}_i$ iff $(U_i\mid\phi)_i = 2\mu_i\int U_i^3\phi$, while $(\phi_1,\phi_2)\in T_{(u,v)}\mathcal{M}$ iff

$$(u\mid\phi_1)_1+(v\mid\phi_2)_2 = 2\mu_1\int u^3\phi_1 + 2\mu_2\int v^3\phi_2+\beta\int u\phi_1v^2+\beta\int u^2v\phi_2.$$

Thus $(\phi_1,\phi_2)\in T_{(U_1,0)}\mathcal{M}$, resp. $T_{(0,U_2)}\mathcal{M}$, iff $(U_i\mid\phi_i)_i=2\mu_i\int U_i^3\phi_i$, proving that

$$(\phi_1,\phi_2)\in T_{(U_1,0)}\mathcal{M}\Leftrightarrow\phi_1\in T_{U_1}\mathcal{N}_1, \qquad (\phi_1,\phi_2)\in T_{(0,U_2)}\mathcal{M}\Leftrightarrow\phi_2\in T_{U_2}\mathcal{N}_2. \tag{5}$$

The next lemma is the key ingredient of our arguments.

Lemma 4. (i) $\forall\beta<\Lambda$, $(U_1,0)$ *and* $(0,U_2)$ *are strict local minima of* Φ *on* $\mathcal{M}$.
(ii) $\forall\beta>\Lambda'$, $(U_1,0)$ *and* $(0,U_2)$ *are saddle points of* Φ *on* $\mathcal{M}$.

Proof. (i) Taking into account that $\Phi'(U_1,0)=\Phi'(0,U_2)=0$, one has

$$\Phi''(U_1,0)[\phi_1,\phi_2]^2 = I_1''(U_1)[\phi_1]^2+\|\phi_2\|_2^2-\beta\int U_1^2\phi_2^2, \quad (\phi_1,\phi_2)\in T_{(U_1,0)}\mathcal{M}.$$

It is well known that U_1 is a minimum of I_1 on $\mathcal{N}_1$. Precisely, there is $c_1>0$ such that

$$I_1''(U_1)[\phi]^2\geqslant c_1\|\phi\|_1^2, \quad \forall\phi\in T_{U_1}\mathcal{N}_1.$$

Then, using (5), we get

$$\Phi''(U_1,0)[\phi_1,\phi_2]^2\geqslant c_1\|\phi_1\|_1^2+\|\phi_2\|_2^2-\beta\int U_1^2\phi_2^2\geqslant c_1\|\phi_1\|_1^2+\|\phi_2\|_2^2-\frac{\beta}{\gamma_1^2}\|\phi_2\|_2^2.$$

Therefore, if $\beta<\gamma_1^2$ there exists $c_2>0$ such that

$$\Phi''(U_1,0)[\phi_1,\phi_2]^2\geqslant c_1\|\phi_1\|_1^2+c_2\|\phi_2\|_2^2.$$

Similarly, if $\beta<\gamma_2^2$, $\exists c_i'>0$ such that

$$\Phi''(0,U_2)[\phi_1,\phi_2]^2\geqslant c_1'\|\phi_1\|_1^2+c_2'\|\phi_2\|_2^2.$$

(ii) Let $\beta>\gamma_1^2$. There exists $\tilde\psi\in H$ such that

$$\gamma_1^2<\frac{\|\tilde\psi\|_2^2}{\int U_1^2\tilde\psi^2}<\beta.$$

According to (5), one has that $(0,\tilde\psi)\in T_{(U_1,0)}\mathcal{M}$. There holds

$$\Phi''(U_1,0)\big[(0,\tilde\psi)\big]^2=\|\tilde\psi\|_2^2-\beta\int U_1^2\tilde\psi^2<0.$$

Similarly, if $\beta>\gamma_2^2$, there exist $\tilde\phi$ such that $\Phi''(0,U_2)[(\tilde\phi,0)]^2<0$. $\square$

Proof of Theorem 1. Let $\beta < \Lambda$. According to Lemma 4(i), we can apply the Mountain Pass theorem to Φ on $\mathcal{M}$ (the Palais–Smale condition is satisfied because H, the space of radial $W^{1,2}(\mathbb{R}^n)$ functions, is compactly embedded in $L^4(\mathbb{R}^n)$ for $n = 2, 3$, see [6]). We get a critical point $(u^*, v^*) \in H \times H$ of Φ and hence a (radial) bound state of (2). Moreover, since $(u^*, v^*) \in \mathcal{M}$, then $(u^*, v^*) \neq (0, 0)$. In order to show that (u^*, v^*) is a positive solution for $\beta > 0$, we consider ($u_+ =$ positive part of u)

$$\tilde{I}_i(u) = \frac{1}{2}\|u\|_i^2 - \frac{1}{4}\mu_i \int u_+^4, \qquad \tilde{\Phi}(u, v) = \tilde{I}_1(u) + \tilde{I}_2(v) - \frac{1}{2}\beta \int u_+^2 v_+^2 \, dx.$$

Of course, the corresponding Nehari manifold $\widetilde{\mathcal{M}}$ is not empty. Indeed, $(U_1, 0), (0, U_2) \in \widetilde{\mathcal{M}}$. Remark that now $(u, v) \mapsto \int u_+^2 v_+^2$ is not C^2. However, it is possible to modify the preceding arguments to show that $\widetilde{\mathcal{M}}$ has the same properties of $\mathcal{M}$. In particular, Lemma 4 still holds yielding, for $\beta < \Lambda$, a Mountain-Pass critical point $(u^*, v^*) \in \widetilde{\mathcal{M}}$ solving

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u_+^3 + \beta u_+ v_+^2, \\ -\Delta v + \lambda_2 v = \mu_2 v_+^3 + \beta u_+^2 v_+. \end{cases}$$

If $\beta > 0$, then $u^* \geqslant 0$, $v^* \geqslant 0$. In addition, one still has that $\Phi(u^*, v^*) > \max\{\Phi(U_1, 0), \Phi(0, U_2)\}$. Let us show that $v^* \not\equiv 0$. Otherwise, $u^* \not\equiv 0$ and from $\Phi'(u^*, 0) = 0$ it follows that u^* is a non-trivial solution of

$$-\Delta u^* + \lambda_1 u^* = \mu_1 (u^*)^3, \quad u^* \in H.$$

Since $u^* \geqslant 0$ and $u^* \not\equiv 0$, then $u^* = U_1$, namely $(u^*, v^*) = (U_1, 0)$, and we get a contradiction to the fact, remarked above, that $\Phi(u^*, v^*) > \Phi(U_1, 0)$. A similar argument proves that $u^* \not\equiv 0$. Since both u^* and v^* are $\not\equiv 0$, using the maximum principle we get $u^* > 0$ and $v^* > 0$. $\square$

Proof of Theorem 2. If $\beta > \Lambda'$, $(U_1, 0)$ and $(0, U_2)$ are saddle points for Φ on $\mathcal{M}$. Therefore $\min_{\mathcal{M}} \Phi$ is achieved at some $(\tilde{u}, \tilde{v})$ such that $0 < \Phi(\tilde{u}, \tilde{v}) < \min\{\Phi(U_1, 0), \Phi(0, U_2)\}$. Repeating the preceding arguments, we find once more that both $\tilde{u} > 0$ and $\tilde{v} > 0$. Using Schwarz symmetrization, it is easy to check that the solution $(\tilde{u}, \tilde{v})$ is minimal among all solutions in $E \times E \setminus \{(0, 0)\}$ and thus is a ground state. $\square$

Remarks 5. (i) If $0 < \beta < \sqrt{\mu_1 \mu_2}$, [5, Theorem 2] contains a result dealing with the existence of solutions (u, v), $u, v > 0$, [5] which are minimal among all the solutions such that *both the components* are positive. They are called in [5] *ground states*, are obtained by a different constrained minimization procedure and could have Morse index 2, like our bound states in Theorem 1. Indeed, we suspect that the latter ones are nothing but the solutions of [5, Theorem 2]. On the other hand, the solutions we found in Theorem 2 are ground states in a sense stronger than [5]: they are minimal among *all the solutions* in $E \times E \setminus \{(0, 0)\}$, have Morse index 1 (in the non-degenerate case) and hence are the natural candidates to give rise to orbitally stable solitary waves. Another new feature of our results is that the bounds Λ, Λ' depend also on the ratio λ_1/λ_2.

(ii) If $\lambda_1 = \lambda_2$ and $\beta \in (0, \Lambda) \cup (\Lambda', +\infty)$, there are explicit solutions of (2). If e.g. $\lambda_1 = \lambda_2 = 1$, they are given by (U_1^*, U_2^*) where

$$U_i^* = \alpha_i U, \quad \alpha_i^2 = \frac{\mu_j - \beta}{\mu_i \mu_j - \beta^2}, \; i \neq j, \; i = 1, 2.$$

We cannot exclude that these solutions coincide with those found in Theorems 1 and 2.

We end this section by giving an idea how one proves the estimates (3). Let

$$\sigma_i^2 \equiv \inf_{\varphi \in E \setminus \{0\}} \frac{\|\varphi\|_i^2}{\|\varphi\|_{L^4}^2} = \inf_{\varphi \in E, \; \|\varphi\|_{L^4} = 1} \|\varphi\|_i^2,$$

denote the best Sobolev constant in the embedding of $(W^{1,2}(\mathbb{R}^n), \| \cdot \|_i)$ into $L^4(\mathbb{R}^n)$. It is easy to see that σ_i is achieved at

$$\hat{v}_i(x) = \sigma_i^{-1} \sqrt{\lambda_i} \, U\big(\sqrt{\lambda_i} \, x\big),$$

5 The proof of this fact is not fully justified in [5] and requires additional arguments.

and one has $\sigma_i^4 = \lambda_i^2 \int U^4(\sqrt{\lambda_i}\,x)\,\mathrm{d}x = \lambda_i^{2-n/2} \int U^4(x)\,\mathrm{d}x$. In particular, there holds

$$\frac{\sigma_j^2}{\sigma_i^2} = \left(\frac{\lambda_j}{\lambda_i}\right)^{1-n/4}.$$

Using this equation, the Hölder inequality and the fact that $\sigma_i^4 = \mu_i^2 \int U_i^4$, we get

$$\gamma_i^2 \geqslant \inf\left[\frac{\|\varphi\|_j^2}{(\int U_i^4)^{1/2}(\int \varphi^4)^{1/2}}\right] \geqslant \frac{\mu_i \sigma_j^2}{\sigma_i^2} = \mu_i \left(\frac{\lambda_j}{\lambda_i}\right)^{1-n/4}.$$

As for the upper bound, we take e.g. $\varphi = U_2$. Then

$$\gamma_1^2 \leqslant \frac{\|U_2\|_2^2}{\int U_1^2 U_2^2} \leqslant \frac{\mu_1\mu_2\|U_2\|_2^2}{\lambda_1\lambda_2 \int U^2(\sqrt{\lambda_1}x)U^2(\sqrt{\lambda_2}x)}.$$

If $\lambda_1 > \lambda_2$ (which implies that $U(\sqrt{\lambda_1}\,x) \leqslant U(\sqrt{\lambda_2}\,x)$), we get

$$\gamma_1^2 \leqslant \frac{\mu_1\mu_2\|U_2\|_2^2}{\lambda_1\lambda_2 \int U^4(\sqrt{\lambda_1}\,x)} = \frac{\lambda_1\mu_1\mu_2^2 \int U_2^4}{\lambda_2\sigma_1^4} = \frac{\lambda_1\mu_1\sigma_2^4}{\lambda_2\sigma_1^4} = \mu_1\left(\frac{\lambda_2}{\lambda_1}\right)^{1-n/2}.$$

Similarly, if $\lambda_1 \leqslant \lambda_2$ then one has $\gamma_1^2 \leqslant \mu_1 \frac{\lambda_2}{\lambda_1}$. Same arguments lead to prove that $\gamma_2^2 \leqslant \max\{\mu_2 \frac{\lambda_1}{\lambda_2}, \mu_2 (\frac{\lambda_2}{\lambda_1})^{1-n/2}\}$.

3. A perturbation result

Motivated by [2], we consider the system (1) in one dimension, with $g(u,v) = uv$. For the sake of simplicity, we will take below $\lambda_i = \mu_i = 1$, $i = 1, 2$. With this notation, the system becomes

$$\begin{cases} -u'' + u = u^3 + \beta v, \\ -v'' + v = v^3 + \beta u. \end{cases} \tag{6}$$

Let $\widehat{U}_\alpha(x) = \frac{\sqrt{2\alpha}}{\cosh(\sqrt{\alpha}\,x)}$ denote the even positive solution of $-u'' + \alpha u = u^3$, $u \in E := W^{1,2}(\mathbb{R})$. For all $\beta \geqslant 0$, (6) has, in addition to the *semi-trivial* solutions $(\pm\widehat{U}_1, 0)$, $(0, \pm\widehat{U}_1)$, two families of soliton like solutions, given by

$$\begin{aligned} &\big(\widehat{U}_{1-\beta}, \widehat{U}_{1-\beta}\big), \big(-\widehat{U}_{1-\beta}, -\widehat{U}_{1-\beta}\big), \quad \text{for } 0 \leqslant \beta \leqslant 1 \quad \text{(symmetric states)}, \\ &\big(\widehat{U}_{1+\beta}, -\widehat{U}_{1+\beta}\big), \big(-\widehat{U}_{1+\beta}, \widehat{U}_{1+\beta}\big), \quad \text{for } \beta \geqslant 0 \quad \text{(anti-symmetric states)}. \end{aligned}$$

The following theorem shows that, for β small, near the anti-symmetric solitons we can find solutions with a quite different profile.

Theorem 6. *If $\beta = \varepsilon > 0$ is sufficiently small, then* (6) *has a solution $(u_\varepsilon, v_\varepsilon) \in E \times E$ such that $u_\varepsilon \sim \widehat{U}_1(x + \xi_\varepsilon) + \widehat{U}_1(x - \xi_\varepsilon)$, $v_\varepsilon \sim -\widehat{U}_1(x)$, where $\xi_\varepsilon \sim -\log\varepsilon$.*

Remarks 7. (i) The first component of the solutions found above has two bumps. In the case of a single NLS equation, the existence of solutions with more than one peak depends on the presence of a suitable potential depending on x. It is a specific remarkable feature of NLS systems that multi-bump solutions exist in the autonomous case as well. Let us also point out that they do not exist near symmetric states, but only near the anti-symmetric ones.

(ii) By using numerical methods, a bifurcation diagram is reported in [2], indicating that for $\beta \in (0,1)$ there exists a family of new solutions, bifurcating from the branch of the anti-symmetric states at $\beta = 1$. Theorem 6 provides a rigorous proof of the existence result, for small values of the parameter β.

The proof of Theorem 6 is carried out by means of perturbation methods. Roughly, one considers the set

$$\mathcal{Z} = \big\{\big(\widehat{U}_1(x+\xi) + \widehat{U}_1(x-\xi), -\widehat{U}_1(x)\big)\colon \xi \in \mathbb{R}, |\xi| \gg 1\big\}.$$

Such a $\mathcal{Z}$ is a manifold of pseudo-critical points of

$$\Phi_\varepsilon(u,v) = I(u) + I(v) - \varepsilon \int_{\mathbb{R}} uv\,\mathrm{d}x, \quad \text{where } I(u) = \frac{1}{2}\int_{\mathbb{R}} \big(|u'|^2 + u^2\big)\,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}} u^4\,\mathrm{d}x,$$

in the sense that Φ_ε' is small on $\mathcal{Z}$, see [4, Section 2.4]. Looking for critical points in the form $z_\xi + w$, with $z_\xi \in \mathcal{Z}$ and $w \perp T_{z_\xi}\mathcal{Z}$, one first finds a unique $w = w_\xi$ which solves the auxiliary equation on $(T_{z_\xi}\mathcal{Z})^\perp$. According to [4, Theorem 2.12] any critical point ξ_ε of the reduced functional $\tilde{\Phi}_\varepsilon(\xi) = \Phi_\varepsilon(z_\xi + w_\xi)$ gives rise to a critical point of Φ_ε and hence to a solution of (6). Carrying out this procedure, one shows that there exist $c_i > 0$, $i = 0, 1, 2$, such that

$$\tilde{\Phi}_\varepsilon(\xi) = 2 + c_0\varepsilon^2 - c_1\,\mathrm{e}^{-2\xi} + \varepsilon c_2 \xi\,\mathrm{e}^{-\xi} + R(\varepsilon, \xi), \tag{7}$$

where $|R(\varepsilon, \xi)| \leqslant \mathrm{o}(\varepsilon)\mathrm{o}(\mathrm{e}^{-2\xi}) + \mathrm{o}(\mathrm{e}^{-4\xi})$, for ε small and $|\xi|$ large, and the theorem follows.

It is easy to extend the preceding arguments to handle the PDE analog of (6). Here one works in $W^{1,2}(\mathbb{R}^n)$, with $n = 2, 3$. It is worth pointing out that now there exist several other multi-bump solutions. More precisely, if $n = 2$, one can prove the existence of solutions with k peaks, for all $k \leqslant 5$, located on a circle of radius $r_\varepsilon \sim -\log\varepsilon$, in the vertices of a regular polygon with k sides. If $n = 3$, the peaks are situated on a sphere with radius as above, in the vertices of any platonic solid, with the exception of the dodecahedron. The reason why the regular polygons with 6 or more sides, resp. the dodecahedron, are excluded is because, in order to obtain an expansion like (7), the side of the polygons, resp. solids, must be smaller than the radius of the circumscribed circle, resp. sphere.

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