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Partial Differential Equations

Diffusion versus absorption in semilinear parabolic problems

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Abstract

We study the limit, when $k \to \infty$, of the solutions $u = u_k$ of (E) $\partial_t u - \Delta u + h(t)u^q = 0$ in $\mathbb{R}^N \times (0,\infty)$, $u_k(\cdot,0) = k\delta_0$, with $q > 1$, $h(t) > 0$. If $h(t) = \mathrm{e}^{-\omega(t)/t}$ where $\omega > 0$ satisfies to $\int_0^1 \sqrt{\omega(t)}\, t^{-1}\,\mathrm{d}t < \infty$, the limit function u_∞ is a solution of (E) with a single singularity at $(0,0)$, while if $\omega(t) \equiv 1$, u_∞ is the maximal solution of (E). We examine similar questions for equations such as $\partial_t u - \Delta u^m + h(t)u^q = 0$ with $m > 1$ and $\partial_t u - \Delta u + h(t)\,\mathrm{e}^u = 0$. ***To cite this article: A. Shishkov, L. Véron, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Diffusion versus absorption dans des problèmes paraboliques semi-linéaires. Nous étudions la limite, quand $k \to \infty$, des solutions $u = u_k$ de (E) $\partial_t u - \Delta u + h(t)u^q = 0$ dans $\mathbb{R}^N \times (0,\infty)$, $u_k(\cdot,0) = k\delta_0$ avec $q > 1$, $h(t) > 0$. Nous montrons que si $h(t) = \mathrm{e}^{-\omega(t)/t}$ où $\omega > 0$ vérifie $\int_0^1 \sqrt{\omega(t)}\, t^{-1}\,\mathrm{d}t < \infty$, la fonction limite u_∞ est une solution of (E) avec une singularité isolée en $(0,0)$, alors que si $\omega(t) \equiv 1$, u_∞ est la solution maximale de (E). Nous examinons des questions semblables pour des équations des type suivants $\partial_t u - \Delta u^m + h(t)u^q = 0$ avec $m > 1$ et $\partial_t u - \Delta u + h(t)\,\mathrm{e}^u = 0$. ***Pour citer cet article : A. Shishkov, L. Véron, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Soit $q > 1$ et $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ une fonction continue, croissante telle que $h(t) > 0$ pour $t > 0$. Il est facile de vérifier que toute solution positive u de

$$\partial_t u - \Delta u + h(t)u^q = 0 \quad \text{dans } \mathbb{R}^N \times]0,+\infty[\tag{1}$$

satisfait à

$$u(x,t) \leqslant U(t) := \left((q-1)\int_0^t h(s)\,\mathrm{d}s\right)^{-1/(q-1)} \qquad \forall (x,t) \in \mathbb{R}^N \times]0,+\infty[. \tag{2}$$

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Si $h \in L^1(0, 1, t^{Nq/2}\,dt)$, il est classique que pour tout $k > 0$ il existe une unique solution (dite fondamentale) $u = u_k$ de (1) sur $\mathbb{R}^N \times]0, +\infty[$ vérifiant $u_k(\cdot, 0) = k\delta_0$. Par le principe du maximum $k \mapsto u_k$ est croissant et deux cas peuvent se produire :

(i) ou bien $u_\infty = \lim_{k\to\infty} u_k = U$. *Explosion initiale complète.*

(ii) ou bien u_∞ est une solution de (1) singulière en $(0, 0)$ vérifiant $\lim_{t\to 0} u_\infty(x, t) = 0$ pour tout $x \neq 0$. *Explosion initiale ponctuelle.*

Théorème 1. (I) *Si $h(t) = e^{-\sigma/t}$ pour un $\sigma > 0$, alors $u_\infty = U$.*

(II) *Si $h(t) = e^{-\omega(t)/t}$ où ω est monotone croissante sur $]0, +\infty[$ et vérifie, pour un $\alpha \in [0, 1[$, $\inf\{\omega(t)/t^\alpha\colon 0 < t \leqslant 1\} > 0$ et $\int_0^1 \sqrt{\omega(t)}\, t^{-1}\,dt < \infty$, alors u_∞ a une explosion initiale ponctuelle.*

Dans le cas de l'équation

$$\partial_t u - \Delta u + h(t)\, e^u = 0 \quad \text{dans } \mathbb{R}^N \times]0, +\infty[, \tag{3}$$

nous montrons.

Théorème 2. (I) *Si $h(t) = e^{-e^{\sigma/t}}$ pour un $\sigma > 0$, alors $u_\infty = \widetilde{U}$.*

(II) *Si $h(t) = e^{-e^{\omega(t)/t}}$ où ω vérifie les conditions du Théorème 1, alors u_∞ a une explosion initiale ponctuelle.*

Nos méthodes nous permettent aussi de traiter l'équation des milieux poreux avec absorption.

1. Main results

Let $q > 1$ and $h : (0, \infty) \mapsto (0, \infty)$ be a continuous nondecreasing function. It is easy to prove that any positive solution u of

$$\partial_t u - \Delta u + h(t) u^q = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty) \tag{1}$$

verifies

$$u(x, t) \leqslant U(t) := \left((q-1) \int_0^t h(s)\,ds \right)^{-1/(q-1)} \qquad \forall (x, t) \in \mathbb{R}^N \times (0, \infty). \tag{2}$$

If $h \in L^1(0, 1, t^{Nq/2}\,dt)$, it is classical that, for any $k > 0$, there exists a unique solution (called fundamental) $u = u_k$ of (1) on $\mathbb{R}^N \times (0, \infty)$ such that $u_k(\cdot, 0) = k\delta_0$. By the maximum principle $k \mapsto u_k$ is increasing and the following alternative occurs:

(i) either $u_\infty = \lim_{k\to\infty} u_k = U$. *Complete initial blow-up.*

(ii) or u_∞ is a solution of (1) singular at $(0, 0)$ such that $\lim_{t\to 0} u_\infty(x, t) = 0$ for all $x \neq 0$. *Single-point initial blow-up.*

Theorem 1. (I) *If $h(t) = e^{-\sigma/t}$ for some $\sigma > 0$, then $u_\infty = U$.*

(II) *If $h(t) = e^{-\omega(t)/t}$ where ω is nondecreasing on $(0, +\infty)$ and verifies, for some $\alpha \in [0, 1)$, $\inf\{\omega(t)/t^\alpha\colon 0 < t \leqslant 1\} > 0$ and*

$$\int_0^1 \frac{\sqrt{\omega(t)}\,dt}{t} < +\infty, \tag{3}$$

then u_∞ has single-point initial blow-up.

Concerning equation

$$\partial_t u - \Delta u + h(t)\, e^u = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty), \tag{4}$$

any solution u verifies

$$u(x,t) \leqslant \widetilde{U}(t) := -\ln\left(\int_0^t h(s)\,\mathrm{d}s\right) \quad \forall (x,t) \in \mathbb{R}^N \times (0,+\infty) \tag{5}$$

and the existence of a fundamental solution $u = u_k$ is ensured if $h(t) = \mathrm{e}^{-b(t)}$ where $\lim_{t\to+\infty} t^{N/2} b(t) = +\infty$.

Theorem 2. (I) *If* $h(t) = \mathrm{O}(\mathrm{e}^{-\mathrm{e}^{\sigma/t}})$ *for some* $\sigma > 0$, *then* $u_\infty = \widetilde{U}$.
(II) *If* $h(t) = \mathrm{e}^{-\mathrm{e}^{\omega(t)/t}}$ *where* ω *satisfies the conditions of Theorem* 1, *then* u_∞ *has single-point initial blow-up.*

Our methods apply to equations of porous media type

$$\partial_t u - \Delta u^m + h(t) u^q = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty), \tag{6}$$

with $m > 1$, $q > 1$ and $h : (0,\infty) \mapsto (0,\infty)$ is nondecreasing. As above, any positive solution satisfies (2). If $h \in L^1((0,1; t^{-(q-1)/(m-1+2N^{-1})}\,\mathrm{d}t)$, for any $k > 0$ there exists a solution $u = u_k$ of (6) such that $u_k(\cdot,0) = k\delta_0$. Since $k \mapsto u_k$ is increasing, the same alternative as in case of (1) occurs concerning u_∞.

Theorem 3. *Assume* $q > m > 1$. (I) *If* $h(t) = \mathrm{O}(t^{(q-m)/(m-1)})$, *then* $u_\infty = U$.
(II) *If* $h(t) = t^{(q-m)/(m-1)}\omega^{-1}(t)$ *where* ω *is nondecreasing and positive on* $(0,+\infty)$ *and verifies*

$$\int_0^1 \frac{\omega^\theta(t)\,\mathrm{d}t}{t} < +\infty, \tag{7}$$

where $\theta = \frac{m^2-1}{(N(m-1)+2(m+1))(q-1)}$, *then* u_∞ *has single-point initial blow-up.*

Sketch of the proofs. The complete initial blow-up results are proved by constructing local subsolutions by modifying the very singular solutions of some related equations. Since for Eq. (1), the proof is already given in [3] we shall outline the (more complicated) construction for Eq. (4).

Lemma 4. *If* $h(t) = \sigma t^{-2} \mathrm{e}^{\sigma t^{-1} - \mathrm{e}^{-\sigma/t}}$ *for some* $\sigma > 0$, *complete initial blow-up occurs for Eq.* (4).

Proof. Writing $h(t) = \mathrm{e}^{-a(t)}$ is first observed that fundamental solutions u_k of (4) exist for all $k > 0$ if $\lim_{t\to 0} t^{N/2} a(t) = \infty$. For $\ell > 1$, let $v = v_{\infty,\ell}$ be the very singular solution of

$$\partial_t v - \Delta v + c t^{\alpha_\ell} v^\ell = 0 \tag{8}$$

in $\mathbb{R}^N \times (0,\infty)$, where α_ℓ and c are positive constants. The choice of $\alpha_\ell = (N+2)/(\ell-1)/2 - 1$ ensures the existence of $v_{\infty,\ell}$. Furthermore, if we write

$$v_{\infty,\ell}(x,t) = \left(\frac{2c}{N+2}\right)^{1/(\ell-1)} t^{-(1+N/2)} f_\ell\big(x/\sqrt{t}\big),$$

then $f_\ell(\eta) \leqslant 1$ for $\eta \in \mathbb{R}^N$ and

$$\Delta f_\ell + \frac{1}{2} D f_\ell \cdot \eta + \frac{N+2}{2} f_\ell - f_\ell^\ell = 0. \tag{9}$$

By the maximum principle $0 < f_\ell < f_{\ell'} \leqslant 1$ for $\ell' > \ell > 1$. For the particular choice $\ell^* = (N+4)/(N2)$, we can use the expression of the asymptotic expansion of the very singular solution given in [1],

$$f_{\ell^*}(\eta) = C|\eta|^2 \mathrm{e}^{-|\eta|^2/4}\big(1 + \mathrm{o}(1)\big) \quad \text{as } |\eta| \to \infty,$$

from which follows $f_\ell(\eta) \geqslant f_{\ell^*}(\eta) \geqslant \delta^*(|\eta|^2 + 1)\,\mathrm{e}^{-|\eta|^2/4}$ for some $\delta^* > 0$, any $\eta \in \mathbb{R}^N$ and $\ell \geqslant \ell^*$. Thus there exists $\delta > 0$ depending only on N such that

$$v_{\infty,\ell}(x,t) \geqslant \delta c^{1/(\ell-1)} t^{-1-N/2}\big(|x|^2 + t\big)\,\mathrm{e}^{-|x|^2/4t} \quad \forall (x,t) \in \mathbb{R}^N \times (0,\infty). \tag{10}$$

Because any positive solution u of (4) satisfies (5), we have to prove that we can fix c and $\tau > 0$ such that

$$ct^{\alpha_\ell}(\rho^\ell + 1) \geqslant h(t)\,\mathrm{e}^{\rho} \quad \forall (t,\rho) \in (0,\tau] \times [0, \widetilde{V}(t)]. \tag{11}$$

Writing h under the form $h(t) = -\omega'(t)\,\mathrm{e}^{\omega(t)}$ where $\omega(t) = \mathrm{e}^{\gamma(t)}$ and γ is a positive decreasing C^1 function, infinite at $t = 0$, we first notice that it is sufficient to prove this inequality for $\rho = \widetilde{U}(t)$, and in that case

$$ct^{\alpha_\ell}\big(\mathrm{e}^{\ell\gamma(t)} + 1\big) \geqslant -\gamma'(t)\,\mathrm{e}^{\gamma(t)} \quad \forall t \in (0,\tau]. \tag{12}$$

We take now $\gamma(t) = \sigma/t$, and prove that there exists $\beta > 0$, depending only on N such that, for any $0 < \tau \leqslant \beta\sigma$, estimate (11) holds with

$$c = \mathrm{e}^{(1-\ell)\sigma/\tau - 2^{-1}(\ell(N+2)-N)\ln\tau}.$$

The maximum principle and (11) imply that for any $\ell > 1$ and $k > 0$ the solutions $u = u_k$ of (4) and $v = \tilde{v}_k$ of

$$\partial_t v - \Delta v + ct^{\alpha_\ell}\big(v^\ell + 1\big) = 0$$

with initial data $k\delta_0$ verifies $0 \leqslant \tilde{v}_{k,\ell} \leqslant u_k$, on $(0,\tau]$. Therefore $v_{\infty,\ell} \leqslant u_\infty + ct^{\alpha_\ell+1}/(\alpha_\ell + 1)$ on $(0,\tau]$ leads to

$$u_\infty(x,\tau) \geqslant \delta\tau^{-1-N/2}\big(|x|^2 + \tau\big)\,\mathrm{e}^{\frac{4\sigma - |x|^2}{4\tau} - \frac{\ell(N+2)-N}{2(1-\ell)}\ln\tau}.$$

Thus $\lim_{\tau\to 0} u_\infty(x,\tau) = \infty$, locally uniformly in $B_{2\sqrt{\sigma}}$, which implies the result. □

The proof of Theorem 2 follows from the fact that for any $\sigma > \sigma' > 0$ there exists an interval $(0,\theta]$ where $\sigma' t^{-2}\,\mathrm{e}^{\sigma' t^{-1} - \mathrm{e}^{-\sigma'/t}} \geqslant \mathrm{e}^{-\mathrm{e}^{\sigma/t}}$.

The single-point initial blow-up is proved by local energy methods (see [2,5,6] for other types of applications). Because of their high degree of technicality we shall just give a short sketch of them in the simplest case of Theorem 1. For $k > 0$, let $u_k = u$ be the solution of the next result.

$$\begin{cases} \partial_t u - \Delta u + h(t)|u|^{q-1}u = 0 & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_{0,k}(x) = M_k^{1/2} k^{-N/2}\eta_k(x) & \forall x \in \mathbb{R}^N, \end{cases} \tag{13}$$

where $\eta_k \in C(\mathbb{R}^N)$ is nonnegative, has compact support in $B_{k^{-1}}$, converges weakly to δ_0 as $k \to \infty$, and $\{M_k\}$ satisfies $\lim_{k\to\infty} k^{-N/2}M_k = \infty$. Furthermore it can be assumed that $\|\eta_k\|_{L^2} \leqslant c_0 k^{N/2}$. The single-point initial blow-up will be a consequence of

Lemma 5. *For any $\delta > 0$ there exists $C = C(\delta)$ such that*:

$$\sup_{t\in[0,1]} \int_{|x|\geqslant\delta} u_k^2(x,t)\,\mathrm{d}x + \int_0^1 \int_{|x|\geqslant\delta} \big(|\nabla u_k|^2 + u_k^2\big)\,\mathrm{d}x\,\mathrm{d}t \leqslant C \quad \forall k \in \mathbb{N}. \tag{14}$$

Proof. For $r \in (0,1)$, $\tau \geqslant 0$ we set $\Omega(\tau) = \{x \in \mathbb{R}^N\colon |x| > \tau\}$, $Q^r(\tau) = \Omega(\tau) \times (0,r]$, $Q_r(\tau) = \Omega(\tau) \times (r,1)$ and $Q_r = \mathbb{R}^N \times (r,1)$, and denote $I_1(r) = \iint_{Q_r} |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t$, $I_2(r) = \iint_{Q_r} u^2\,\mathrm{d}x\,\mathrm{d}t$, $I_3(r) = \iint_{Q_r} h(t)|u|^{q+1}\,\mathrm{d}x\,\mathrm{d}t$. If we multiply the equation by $u(x,t)\,\mathrm{e}^{(r-t)/(2-r)}$, integrate on Q_r and use Hölder's inequality, we get,

$$\begin{aligned} \int_{\mathbb{R}^N} u^2(x,1)\,\mathrm{d}x + I_1(r) + I_2(r) + I_3(r) &\leqslant c\int_{\mathbb{R}^N} u^2(x,r)\,\mathrm{d}x \\ &\leqslant c\tau^{\frac{N(q-1)}{q+1}} h(r)^{\frac{-2}{q+1}}\big(-I_3'(r)\big)^{\frac{2}{q+1}} + c\int_{\Omega(\tau)} u^2(x,r)\,\mathrm{d}x. \end{aligned} \tag{15}$$

Let $\tau \mapsto \mu(\tau)$ be a smooth decreasing function, we define

$$E_1^{\mu}(r,\tau)=\iint\limits_{Q^r(\tau)}\big(|\nabla u|^2+\mu^2u^2(x,t)\big)\,\mathrm{e}^{-\mu^2t}\,\mathrm{d}x\,\mathrm{d}t,$$

$$E_2(r,\tau)=\iint\limits_{Q^r(\tau)}u^2\,\mathrm{d}x\,\mathrm{d}t\quad\text{and}\quad f_\mu(r,\tau)=\sup\Big\{\mathrm{e}^{-\mu^2t}\int\limits_{\Omega(\tau)}u^2(x,t)\,\mathrm{d}x\colon\ 0\leqslant t\leqslant r\Big\}$$

and $f(r)=f_0(r,0)$. Then we introduce a parameter in the equation as in [4] by multiplying it by $u(x,t)\exp(-\mu^2(\tau)t)$ and integrating in the domain $Q^r(\tau)$ with $\tau>k^{-1}$ $Q^r(\tau)$ and $\tau>k^{-1}$. After some simple computations we deduce:

$$f_\mu(r,\tau)+2E_1^{\mu}(r,\tau)\leqslant\frac{2}{\mu}\int\limits_0^r\int\limits_{\partial\Omega(\tau)}\big(|\nabla u|^2+\mu^2u^2(x,t)\big)\,\mathrm{e}^{-\mu^2t}\,\mathrm{d}S\,\mathrm{d}t\quad\forall\tau>k^{-1}.$$

Assuming $1-2\mu'/\mu^2>1/2$, we deduce from last inequality:

$$f_\mu(r,\tau)+E_1^{\mu}(r,\tau)\leqslant-\frac{2}{\mu(\tau)}\frac{\mathrm{d}E_1^{\mu}(r,\tau)}{\mathrm{d}\tau}\quad\forall\tau>k^{-1},$$

and by integration

$$f_\mu(r,\tau_2)\big(\mathrm{e}^{\int_{\tau_1}^{\tau_2}\frac{\mu(\tau)\,\mathrm{d}\tau}{2}}-1\big)+E_1^{\mu}(r,\tau_2)\,\mathrm{e}^{\int_{\tau_1}^{\tau_2}\frac{\mu(\tau)\,\mathrm{d}\tau}{2}}\leqslant E_1^{\mu}(r,\tau_1)\quad\forall\tau_2>\tau_1>k^{-1}.$$

The choice $\mu(\tau)=r^{-1}(\tau-k^{-1})/8$ $(\tau>k^{-1})$ yields to

$$\begin{aligned}&\int\limits_{\Omega(\tau)}u^2(x,r)\,\mathrm{d}x+\iint\limits_{Q^r(\tau)}\Big(|\nabla_xu|^2+\frac{(\tau-k^{-1})^2}{64r^2}u^2\Big)\,\mathrm{d}x\,\mathrm{d}t\\&\quad\leqslant c_1\,\mathrm{e}^{-\frac{(\tau-k^{-1})^2}{64r}}\times\iint\limits_{Q^r(\tau_0^k)}\Big(|\nabla_xu|^2+\frac{u^2}{2r}\Big)\,\mathrm{d}x\,\mathrm{d}t\quad\forall\tau\geqslant\tilde\tau_0^k:=k^{-1}+8\sqrt{r}>\tau_0^k:=k^{-1}+4\sqrt{2r}.\end{aligned}\tag{16}$$

We will need standard global energy estimate of solution of problem (13) too:

$$\int\limits_{\mathbb{R}^N}\big|u(x,r)\big|^2\,\mathrm{d}x+\int\limits_{Q^r}\big(|\nabla_xu|^2+|u|^2+h(t)|u|^{q+1}\big)\,\mathrm{d}x\,\mathrm{d}t\leqslant c\|u_{0,k}\|^2_{L_2(\mathbb{R}^N)}\leqslant\bar cM_k\quad\forall r>0.\tag{17}$$

Estimating the right-hand side terms in (15) and (16) by (17), we derive:

$$\begin{aligned}&\text{(i)}\ \sum_{i=1}^3I_i(r)\leqslant c_1\tau^{\frac{N(q-1)}{q+1}}h(r)^{\frac{-2}{q+1}}\big(-I_3'(r)\big)^{\frac{2}{q+1}}+\frac{c_2M_k}{r}\,\mathrm{e}^{-(\tau-k^{-1})^2/64r}\quad\forall\tau\geqslant\tilde\tau_0^k(r),\\&\text{(ii)}\ f_0(r,\tau)+E_1^0(r,\tau)+\frac{\tau-k^{-1}}{64r^2}E_2(r,\tau)\leqslant\frac{c_2M_k}{r}\,\mathrm{e}^{-(\tau-k^{-1})^2/64r}\quad\forall\tau\geqslant\tilde\tau_0^k(r).\end{aligned}\tag{18}$$

Next we choose $M_k=\mathrm{e}^{\mathrm{e}^k}$, fix $\epsilon_0\in(0,\mathrm{e}^{-1})$ and define a pair (r_k,τ_k) by the following relations: $r_k=\sup\{r\colon\ I_1(r)+I_2(r)+I_3(r)>2M_k^{\epsilon_0}\}$; $c_2r_k^{-1}\exp(-\frac{\tau_k^2}{64r_k})M_k=M_k^{\epsilon_0}\Leftrightarrow\tau_k=8\sqrt{r_k(1-\epsilon_0)\,\mathrm{e}^k+\ln(c_2/r_k)}$. Taking $\tau=\tau_k+k^{-1}$ in (18)(i) and solving the corresponding O.D.E. yields the estimate:

$$\sum_{i=1}^3I_i(r)\leqslant c_3\big(\tau_k+k^{-1}\big)\big(H(r)\big)^{-2/(q-1)}\quad\forall r\leqslant r_k,\ H(r)=\int\limits_0^rh(s)\,\mathrm{d}s.\tag{19}$$

If we write $h(t)=\mathrm{e}^{-\omega(t)/t}$, the assumption $\inf\{\omega(t)/t^{\alpha}\colon\ 0<t\leqslant1\}>0$ implies that $H(r)\geqslant c_0\,\mathrm{e}^{-\omega(r)/r}r^2/\omega(r)$ and, replacing τ_k by its expression, (19) turns into

$$\sum_{i=1}^3I_i(r)\leqslant c_4\Big(\sqrt{r_k(1-\epsilon_0)\,\mathrm{e}^k+\ln(c_2/r_k)}+k^{-1}\Big)^N\Big(\frac{\omega(r)\mathrm{e}^{-\omega(r)/r}}{r^2}\Big)^{2/(q-1)}\quad\forall r\leqslant r_k.$$

Thus $r_k \leqslant b_k$, where b_k is solution of equation:

$$c_4\Big(\sqrt{r_k(1-\epsilon_0)\,\mathrm{e}^k+\ln(c_2/b_k)}\,+k^{-1}\Big)^N\left(\frac{\omega(b_k)\,\mathrm{e}^{-\omega(b_k)/b_k}}{b_k^2}\right)^{2/(q-1)}=2M_k^{\epsilon_0}=2\mathrm{e}^{\epsilon_0\mathrm{e}^k}.$$

From this inequality, using additionally assumption on $\omega(t)$, we obtain: $c_5\mathrm{e}^k \geqslant \omega(b_k)/b_k \geqslant c_6\,\mathrm{e}^k$, $c_6>0$; $b_k \geqslant \mathrm{e}^{-c_7k}$, $c_7>0$. These inequalities yield:

$$\tau_k \leqslant c_8\sqrt{\omega\big(c_9\,\mathrm{e}^{-k}\big)}. \tag{20}$$

Using the definition of r_k, inequality (18)(ii), the fact that $3M_k^{\epsilon_0} \leqslant \bar{c}M_{k-1}$ $\forall k \geqslant k_0(\bar{c})$ ($\bar{c}$ is from (17), $0<\epsilon_0<\mathrm{e}^{-1}$), we deduce the main result of first round of computations:

$$\sum_{i=1}^{3} I_i(r_k)+f_0\big(r_k,\tau_k+k^{-1}\big)+\sum_{i=1}^{2} E_i\big(r_k,\tau_k+k^{-1}\big) \leqslant 3M_k^{\epsilon_0} \leqslant \bar{c}M_{k-1}. \tag{21}$$

Next we organize the second round of estimates with $\mu(\tau)=(\tau-\tau_k-k^{-1})/8$, r_{k-1} and τ_{k-1} be defined similarly as r_k and τ_k, up to the change of indices, using obtained estimate (21) instead of (17). As result we derive:

$$\sum_{i=1}^{3} I_i(r_{k-1})+f_0\big(r_{k-1},\tau_k+\tau_{k-1}+k^{-1}\big)+\sum_{i=1}^{2} E_i\big(r_{k-1},\tau_k+\tau_{k-1}+k^{-1}\big) \leqslant \bar{c}M_{k-2}. \tag{22}$$

Fixing arbitrary $n>k_0(\bar{c})$ and repeating the above described round of computations $k-n$ times, we obtain:

$$\sum_{i=1}^{3} I_i(r_n)+f_0\left(r_n,\sum_{j=0}^{k-n}\tau_{k-j}+k^{-1}\right)+\sum_{i=1}^{2} E_i\left(r_n,\sum_{j=0}^{k-n}\tau_{k-j}+k^{-1}\right) \leqslant \bar{c}M_{n-1}, \tag{23}$$

and, since by induction τ_{k-j} satisfies (20) with k replaced by $k-j$, we obtain

$$\sum_{j=0}^{k-n}\tau_{k-j} \leqslant c_8\sum_{j=0}^{k-n}\sqrt{\omega\big(c_9\,\mathrm{e}^{-(k-j)}\big)} \leqslant c_{10}\int_{c_9\,\mathrm{e}^{-k}}^{c_9\,\mathrm{e}^{-n}}\frac{\sqrt{\omega(s)}\,\mathrm{d}s}{s}. \tag{24}$$

We denote $\tau^*(n)=\lim_{k\to\infty}c_8\sum_{j=0}^{k-n}\sqrt{\omega(c_9\,\mathrm{e}^{-(k-j)})}$. We derive from (23) by letting $k\to\infty$,

$$\sup_{0<t\leqslant r_n}\int_{|x|\geqslant\tau^*(n)} u^2(x,t)\,\mathrm{d}x+\int_0^{r_n}\int_{|x|\geqslant\tau^*(n)}\big(|Du|^2+u^2\big)\,\mathrm{d}x\,\mathrm{d}t \leqslant \bar{c}M_{n-1}. \tag{25}$$

Due to assumption (4) $\tau^*(n)\to 0$ as $n\to\infty$, therefore inequality (25) implies the result.

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