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Dynamical Systems

Lipschitz equivalence of self-similar sets

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Abstract

In 1997 David and Semmes asked whether there exists a bilipschitz map between the two compact self-similar subset M and M' of the real line defined by the relations $M=(M/5)\cup(M/5+2/5)\cup(M/5+4/5)$ and $M'=(M'/5)\cup(M'/5+3/5)\cup(M'/5+4/5)$. We answer this question positively. ***To cite this article: H. Rao et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Equivalence Lipschitz d'ensembles autosimilaires. En 1997, David et Semmes ont posé la question de savoir s'il existe une application bi-lipschitzienne entre les deux compacts linéaires M et M' définis par les relations $M=(M/5)\cup(M/5+2/5)\cup(M/5+4/5)$ et $M'=(M'/5)\cup(M'/5+3/5)\cup(M'/5+4/5)$. Nous répondons affirmativement à cette question. ***Pour citer cet article : H. Rao et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

On dira que deux espaces métriques A et B sont Lipschitz-équivalents s'il existe une application bi-lipschitzienne de l'un sur l'autre. Ce travail a été motivé par la question rappelée dans le résumé. La réponse est, de façon un peu surprenante, positive.

Nous nous plaçons dans un cadre plus général que celui de la question posée. Etant donnés un entier $n\geqslant 2$ et $r\in\,]0,1/n[$, on désigne par $\mathcal{S}_{n,r}$ l'ensemble des compacts F de $\mathbb{R}$ tels que

(i) $F=\bigcup_{i=1}^{n}S_i(F)$, où $S_i(x)=rx+b_i$,
(ii) $0=b_1<b_2<\cdots<b_n=1-r$,
(iii) si $i\neq j$, alors $S_i([0,1])\cap S_j([0,1])$ a au plus un point.

On a alors le résultat suivant :

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Théorème 0.1. *Tous les éléments de $\mathcal{S}_{n,r}$ sont Lipschitz-équivalents.*

La démonstration utilise la notion de 'graph directed sets' de [6] que nous rappelons. On se donne un graphe orienté $G=(V,\Gamma)$ et pour chaque arête $e\in\Gamma$ une contraction affine T_e de $\mathbb{R}$. On sait qu'alors il existe une unique famille $\{E_i\}_{i\in V}$ de compacts non vides de $\mathbb{R}$ telle que l'on ait, pour tout $i\in V$,

$$E_i=\bigcup_{j\in V}\bigcup_{e\in E_{i,j}}T_e(E_j),\tag{1}$$

où $E_{i,j}$ désigne l'ensemble des arêtes joignant le sommet i au sommet j.

Dans le cas où toutes les réunions (1) sont disjointes, nous dirons comme dans [4] que les ensembles E_i sont des poussières. Nous avons alors le résultat suivant :

Théorème 0.2. *Soit $\{E_i\}_{i\in V}$ et $\{F_i\}_{i\in V}$ deux poussières construites au moyen du même graphe (V,E). Si, pour chaque $e\in E$, les applications associées ont le même rapport de contraction, alors, pour tout $i\in V$, les ensembles E_i et F_i sont Lipschitz-équivalents.*

La démonstration consiste à identifier E_i et F_i aux chemins du graphe issus du sommet i.

Dans cette version abrégée, nous donnons la démonstration du Théorème 0.1 seulement dans le cas envisagé dans le résumé. Soit M et M' les ensembles définis par

$$M=(M/5)\cup(M/5+2/5)\cup(M/5+4/5),\qquad M'=(M'/5)\cup(M'/5+3/5)\cup(M'/5+4/5).\tag{2}$$

Considérons les ensembles $M_1=M$, $M_2=M\cup(M+2)$ et $M_3=M\cup(M+2)\cup(M+4)$. Il résulte de la formule (2) que l'on a

$$\begin{aligned}
&M_1=M/5\cup(M/5+2/5)\cup(M/5+4/5)=M_1/5\cup(M_2/5+2/5),\\
&M_2=(M_1/5+2)\cup(M_3/5)\cup(M_2/5+12/5),\\
&M_3=(M_1/5+4)\cup(M_3/5)\cup(M_3/5+2)\cup(M_2/5+22/5).
\end{aligned}\tag{3}$$

Par suite, les ensembles $\{M_1,M_2,M_3\}$ sont les ensembles invariants d'une itération de fonctions dirigée par le graphe G représenté par la Fig. 1, les rapports de similitudes valant tous $1/5$.

On fait de même avec M' : on pose $M'_1=M'$, $M'_2=M'\cup(M'+1)$ et $M'_3=M'\cup(M'+1)\cup(M'+2)\subset[0,3]$. On a

$$\begin{aligned}
&M'_1=M'_1/5\cup(M'_2/5+3/5),\\
&M'_2=M'_1/5\cup(M'_3/5+3/5)\cup(M'_2/5+8/5),\\
&M'_3=M'_1/5\cup(M'_3/5+3/5)\cup(M'_3/5+8/5)\cup(M'_2/5+13/5).
\end{aligned}\tag{4}$$

Ainsi, on peut construire les ensembles M'_1, M'_2 et M'_3 au moyen d'un système dirigé par le même graphe que précédemment.

Il est facile de montrer que les ensembles $\{M_i\}_{i=1}^3$ et $\{M'_i\}_{i=1}^3$ sont des poussières. On conclut au moyen du Théorème 0.2.

1. Introduction

Two compact metric spaces (A,d_A) and (B,d_B) are *Lipschitz equivalent*, and then we write $A\simeq B$, if there is a bijection f from A to B and a constant $C>0$ such that, for all $a_1,a_2\in A$, we have

$$C^{-1}\mathrm{d}_A(a_1,a_2)\leqslant\mathrm{d}_B\big(f(a_1),f(a_2)\big)\leqslant C\,\mathrm{d}_A(a_1,a_2).\tag{5}$$

It is well-known that $A\simeq B$ implies $\dim_H A=\dim_H B$, where $\dim_H$ denotes the Hausdorff dimension. Falconer and Marsh [3] proved an interesting result which says that two quasi-self-similar circles have the same Hausdorff dimension if and only if they are Lipschitz equivalent.

Recently, many works have been devoted to the study of the Lipschitz equivalence between self-similar sets ([1,2,4, 8,9]...). Falconer and Marsh [4] obtained some necessary conditions for two dust-like self-similar sets to be Lipschitz equivalent. In [9], the third author proved that dust-like self-conformal sets have the same Hausdorff dimension if and only if they are nearly Lipschitz equivalent. Wen and Xi [8] constructed self-similar arcs with the same Hausdorff dimension but not Lipschitz equivalent.

Indeed, very little is known concerning the Lipschitz equivalence between self-similar sets. It can be seen from an open question in [2]: Problem 11.16 of [2] asks whether the two self-similar sets M and M' defined by (2) are Lipschitz equivalent. The authors of [2] expected a negative answer. However, we have the following results:

Proposition 1.1. Let M and M' be the self-similar sets defined by (2), then $M \simeq M'$.

We can generalize this result to a more general setting. Let $n \geqslant 2$ and $0 < r < 1/n$. Denote by $\mathcal{S}_{n,r}$ the collection of self-similar sets F satisfying:

(i) $F = \bigcup_{i=1}^{n} S_i(F)$, where $S_i(x) = rx + b_i$;
(ii) $0 = b_1 < b_2 \cdots < b_n = 1 - r$;
(iii) The intersection $S_i[0,1] \cap S_j[0,1]$ contains at most one point for $i \neq j$.

Then we have

Theorem 1.2. *For any self-similar sets E and F in $\mathcal{S}_{n,r}$, we have $E \simeq F$.*

This Note is organized as follows. In Section 2, we study the Lipschitz equivalence of graph-directed sets. Proposition 1.1 is proven in Section 3, and Theorem 1.2 in the last section.

2. Lipschitz equivalence of dust-like graph-directed sets

Let $\{f_i\}_{i=1}^N$ be a family of contractive similitudes in $\mathbb{R}^n$. Then there is a unique non-empty compact set E — called the *invariant set* or *self-similar set* of the system $\{f_i\}_{i=1}^N$ — such that $E = \bigcup_{i=1}^N f_i(E)$ [5]. If $f_i(E) \cap f_j(E) = \varnothing$ for all $i \neq j$, the self-similar set E is said to be *dust-like*.

Let $\{f_i\}_{i=1}^N$ and $\{g_i\}_{i=1}^N$ be two families of contractive similitudes whose invariant sets are E and F. Suppose E and F are dust-like, and the contraction ratios of f_i and g_i are equal for all $1 \leqslant i \leqslant N$. Then it is easy to show that E and F are Lipschitz equivalent. In this section, we generalize this result to graph-directed sets, which play an important rôle in our study.

We first recall the definition of graph-directed sets [6]. Let $G = (V, \Gamma)$ be a directed graph with vertex set $V = \{1, \ldots, N\}$ and directed-edge set Γ. We assume that for any $1 \leqslant i \leqslant N$, there is at least one edge starting from vertex i. For any edge $e \in \Gamma$, there is a corresponding similitude $T_e : \mathbb{R}^n \to \mathbb{R}^n$ with similarity ratio $\rho_e \in (0,1)$. That is, $|T_e(x) - T_e(y)| = \rho_e |x - y|$, $\forall x, y \in \mathbb{R}^n$. The graph G labelled with the similitudes T_e will be denoted by G^*. We call G the base graph of G^*. The set of edges from i to j is denoted by $\Gamma_{i,j}$. The *graph-directed sets on G^** are the unique non-empty compact sets $\{E_i\}_{i=1}^N$ satisfying

$$E_i = \bigcup_{j=1}^{N} \bigcup_{e \in \Gamma_{i,j}} T_e(E_j), \quad 1 \leqslant i \leqslant N.$$

We say that $\{E_i\}_{i=1}^N$ are dust-like, if the above union is a disjoint union for each i.

Theorem 2.1. *Let $\{E_i\}_{i=1}^N$ and $\{F_i\}_{i=1}^N$ be the graph-directed sets on G^* and H^* respectively. If*

(i) *The base graphs coincide, i.e., $G = H$*

$$\left(\text{so, we have } E_i = \bigcup_{j=1}^{N} \bigcup_{e \in \Gamma_{i,j}} T_e(E_j) \text{ and } F_i = \bigcup_{j=1}^{N} \bigcup_{e \in \Gamma_{i,j}} S_e(F_j) \right);$$

(ii) *For each edge e of G, the similitudes S_e and T_e have the same ratio ρ_e;*

(iii) $\{E_i\}_{i=1}^N$ *and* $\{F_i\}_{i=1}^N$ *are dust-like.*

Then $E_i \simeq F_i$ *holds for all* $1 \leqslant i \leqslant N$.

Proof. Let $\operatorname{diam} E = \sup\{|x-y|\colon\ x, y \in E\}$ stand for the diameter of E. The distance between two sets X, Y is defined to be $\mathrm{d}(X,Y) = \inf\{|x-y|\colon\ x \in X,\ y \in Y\}$.

Since $\{E_i\}_{i=1}^N$ are dust-like, for any $x \in E_i$, there is a unique infinite path $e_1e_2e_3\cdots$ starting from vertex i such that

$$\{x\} = \bigcap_{k=1}^{\infty} T_{e_1\cdots e_k}(E_{i_k}),$$

where i_k is the ending vertex of edge e_k, and $T_{e_1\cdots e_k} = T_{e_1} \circ \cdots \circ T_{e_k}$. We say that $e_1e_2e_3\cdots$ is a coding of x. Hence the mapping $f : E_i \mapsto F_i$ defined by

$$\bigl\{f(x)\bigr\} = \bigcap_{k=1}^{\infty} S_{e_1\cdots e_k}(F_{i_k})$$

is a bijection. It remains to show that f satisfies (5).

Suppose $x, x' \in E_i$. Let $e_1e_2e_3\cdots$ and $e'_1e'_2e'_3\cdots$ be the codings of x and x'. Let m be the largest integer such that $e_1e_2\cdots e_m = e'_1e'_2\cdots e'_m$. Since both x and x' are in the set $T_{e_1\cdots e_m}(E_{i_m})$, we have

$$|x-x'| \leqslant \operatorname{diam} T_{e_1\cdots e_m}(E_{i_m}) = \left(\prod_{i=1}^{m}\rho_{e_i}\right)\operatorname{diam}(E_{i_m}).$$

On the other hand, by the maximality of m, we have

$$|x-x'| \geqslant \mathrm{d}\bigl(T_{e_1\cdots e_m e_{m+1}}(E_{i_{m+1}}), T_{e_1\cdots e_m e'_{m+1}}(E_{i'_{m+1}})\bigr) \geqslant \left(\prod_{i=1}^{m}\rho_{e_i}\right)\min_{(e,e')}\mathrm{d}\bigl(T_e(E_j), T_{e'}(E_{j'})\bigr),$$

where the minimum is taking over all the pairs (e, e') of distinct edges stemming from a same vertex. For such a pair, let j and j' stand for the ending vertices of e and e'. Since e and e' start from a same vertex i, $T_e(E_j)$ and $T_{e'}(E_{j'})$ are disjoint closed subsets of E_i. Hence the minimum is a positive number.

Therefore, there exists a constant $c_1 > 0$ depending only on G^* such that

$$c_1^{-1}\prod_{i=1}^{m}\rho_{e_i} \leqslant |x-x'| \leqslant c_1\prod_{i=1}^{m}\rho_{e_i}.$$

Similarly, there exists a constant $c_2 > 0$ depending only on H^* such that

$$c_2^{-1}\prod_{i=1}^{m}\rho_{e_i} \leqslant |f(x)-f(x')| \leqslant c_2\prod_{i=1}^{m}\rho_{e_i}.$$

It follows that $c_1^{-1}c_2^{-1}|x-x'| \leqslant |f(x)-f(x')| \leqslant c_1c_2|x-x'|$. □

3. Proof of Proposition 1.1

Recall that M and M' are self-similar sets defined by (2). We will find graph-directed structures for M and M', and then get Proposition 1.1 by Theorem 2.1. The idea of studying graph-directed structures of self-similar sets appeared in [7], where they deal with self-similar sets with overlaps.

Consider the following sets $M_1 = M$, $M_2 = M \cup (M+2)$, and $M_3 = M \cup (M+2) \cup (M+4)$. According to formula (2), M_1 can be subdivided into copies of $M/5$. By regrouping these small copies, one obtains formulas (3). Hence $\{M_1, M_2, M_3\}$ are the invariant sets of a graph-directed system G^*, where the base graph G is depicted by Fig. 1, and the similitudes can be read from the above set equations.

A similar decomposition can be performed for M': $M'_1 = M'$, $M'_2 = M' \cup (M'+1)$, and $M'_3 = M' \cup (M'+1) \cup (M'+2) \subset [0,3]$. Again by formula (2), one obtains

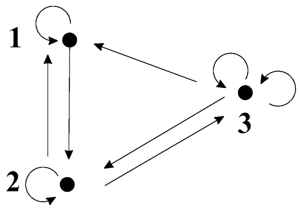

Fig. 1. The directed graph G.

$$M'_1 = M'_1/5 \cup (M'_2/5 + 3/5),$$
$$M'_2 = M'_1/5 \cup (M'_3/5 + 3/5) \cup (M'_2/5 + 8/5),$$
$$M'_3 = M'_1/5 \cup (M'_3/5 + 3/5) \cup (M'_3/5 + 8/5) \cup (M'_2/5 + 13/5).$$

Clearly, the base graph of $\{M'_1, M'_2, M'_3\}$ is still that depicted in Fig. 1.

It is easy to show that $\{M_i\}_{i=1}^3$ and $\{M'_i\}_{i=1}^3$ are dust-like. Since all the similitudes have ratio $1/5$, Theorem 2.1 shows that $M_1 \simeq M'_1$, i.e., $M \simeq M'$.

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2, by using the same idea as in Section 3.

As previously $n \geqslant 2$ and $0 < r < 1/n$. Set $h = (1 - nr)/(n - 1)$. Let E be the self-similar set given by

$$E = \bigcup_{i=1}^{n} \big(rE + (i-1)(r+h)\big). \tag{6}$$

Recall that $\mathcal{S}_{n,r}$ is the collection of self-similar sets $F = \bigcup_{i=1}^n S_i(F)$, where $S_i(x) = rx + b_i$, $0 = b_1 < b_2 < \cdots < b_n = 1 - r$, and $S_i[0,1] \cap S_j[0,1]$ consists of at most one point for $i \neq j$. In the following, we will show that $E \simeq F$ for any $F \in \mathcal{S}_{n,r}$, which yields Theorem 1.2.

Let $K = \bigcup_{i=1}^n S_i[0,1]$. Let $K_1, K_2, \ldots, K_m$ be the connected components of K, numbered from left to right. Let $a_i = |K_i|/r$, where $|K_i|$ is the length of K_i. Then $\sum_{i=1}^m a_i = n$.

For $1 \leqslant i \leqslant n$, set $F_i = F \cup (F+1) \cup \cdots \cup (F+i-1)$. In particular $F_1 = F$. We claim that $\{F_i\}_{i=1}^N$ are dust-like graph-directed sets on G with vertex set $\{1, 2, \ldots, n\}$.

Let $0 = c_1 < c_2 < \cdots < c_m$ be the left endpoints of the intervals $K_1, K_2, \ldots, K_m$. Then

$$F_1 = F = \bigcup_{j=1}^{m} (rF_{a_j} + c_j)$$

and the right-hand side is a disjoint union. We set it to be the set equation for F_1. Then on the graph G, the edges starting from 1 end at $a_1, a_2, \ldots, a_m$ respectively. Notice that several a_i may coincide, which means there may exist several edges from 1 to a_i.

Since F_2 is a union of two copies of F, we can put the set equation of F_2 as follows:

$$F_2 = \left(\bigcup_{j=1}^{m-1} (rF_{a_j} + c_j)\right) \cup (rF_{a_m + a_1} + c_m) \cup \left(\bigcup_{j=2}^{m} (rF_{a_j} + c_j + 1)\right).$$

Again the right-hand side is a disjoint union. On the graph G, the edges starting from 2 end at

$$a_1, a_2, \ldots, a_{m-1}, (a_m + a_1), a_2, \ldots, a_m$$

respectively. In general, we can obtain a set equation of F_k in the same way. The edges of G starting from k end at the vertices

$$a_1, a_2, \ldots, a_{m-1}, \big\{(a_m + a_1), a_2, \ldots, a_{m-1}\big\}^{k-1}, a_m \tag{7}$$

respectively. Obviously $\{F_i\}_{i=1}^n$ are dust-like.

Now we turn to consider another graph-directed system. For $1 \leqslant i \leqslant n$, set

$$E_i = E \cup \big(E + \big(1 + hr^{-1}\big)\big) \cup \cdots \cup \big(E + (i-1)\big(1 + hr^{-1}\big)\big),$$

where E is defined by (6). We shall show that $\{E_i\}_{i=1}^n$ are also graph-directed sets on the graph G.

First we subdivide the set $E_1 = E$ into n copies of rE. Then we group them up according to the structure of F_1 and we get

$$E = (rE_{a_1} + d_1) \cup (rE_{a_2} + d_2) \cup \cdots \cup (rE_{a_{m-1}} + d_{m-1}) \cup (rE_{a_m} + d_m), \tag{8}$$

where $d_i = (a_1 + \cdots + a_{i-1})(r + h)$. We set (8) to be the set equation for E_1. Hence the edges starting from 1 end at vertices $a_1, a_2, \ldots, a_m$ respectively. Since the n copies of rE are uniformly distributed in the interval $[0, 1]$, we may group them up in another way:

$$E = \big(rE_{a_2} + d_2'\big) \cup \big(rE_{a_3} + d_3'\big) \cup \cdots \cup \big(rE_{a_{m-1}} + d_{m-1}'\big) \cup \big(rE_{a_m + a_1} + d_m'\big), \tag{9}$$

where $d_i' = (a_2 + \cdots + a_{i-1})(r + h)$. As the set E_k consists of k translation copies of E it consists of kn translation copies of rE as well. We group the first n copies by formula (8), and group the other $(k-1)n$ copies by formula (9). Then the edges starting from vertex k end at vertices

$$a_1, a_2, \ldots, a_{m-1}, a_m, \big\{(a_m + a_1), a_2, \ldots, a_{m-1}\big\}^{k-1}. \tag{10}$$

The sets of vertices in (7) and (10) coincide. Thus $\{E_i\}_{i=1}^n$ are also graph-directed sets on the graph G.

It is easy to see that both $\{F_i\}$ and $\{E_i\}$ are dust-like. Since all the similitudes have contraction ratio r, by Theorem 2.1, we have $E_1 \simeq F_1$, i.e., $E \simeq F$.

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