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C. R. Acad. Sci. Paris, Ser. I 342 (2006) 229–232

COMPTES RENDUS MATHEMATIQUE

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Algebra

# Universal deformation rings need not be complete intersections ☆

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Received 4 October 2005; accepted 6 December 2005

Available online 4 January 2006

Presented by Jean-Pierre Serre

**Abstract**

We answer a question of M. Flach by showing that there is a linear representation of a profinite group whose universal deformation ring is not a complete intersection. We show that such examples arise in arithmetic in the following way. There are infinitely many real quadratic fields $F$ for which there is a mod 2 representation of the Galois group of the maximal unramified extension of $F$ whose universal deformation ring is not a complete intersection. ***To cite this article: F.M. Bleher, T. Chinburg, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***


**Résumé**

**Les anneaux de déformation universelle ne sont pas nécessairement d'intersection complète.** Nous répondons à une question de M. Flach en démontrant qu'il existe une représentation linéaire d'un groupe profini dont l'anneau de déformation universelle n'est pas un anneau d'intersection complète. Nous montrons que l'arithmétique fournit de tels exemples dans les situations suivantes. Il existe une infinité de corps quadratiques réels $F$ tels qu'il existe une représentation du groupe de Galois de l'extension maximale non-ramifiée de $F$ sur un corps de caractéristique 2 dont l'anneau de déformation universelle n'est pas un anneau d'intersection complète. ***Pour citer cet article : F.M. Bleher, T. Chinburg, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***


## 1. Introduction

In this Note we answer a question of Flach [3] by giving an example of a linear representation of a profinite group over a field of positive characteristic which has an (unrestricted) universal deformation ring which is not a complete intersection. To our knowledge, this is the first example of such a representation. In Section 3 we discuss some arithmetic examples.

## 2. The non-trivial irreducible mod 2 representation of $S_4$

**Theorem 2.1.** *Let $k$ be a perfect field of characteristic 2, let $W$ be the ring of infinite Witt vectors over $k$, and let $S_4$ be the symmetric group on 4 letters. Let $V$ be a non-trivial irreducible $kS_4$-module of dimension 2; $V$ is unique up to*

☆ Supported (respectively) by NSF Grant DMS01-39737 and NSF Grant DMS00-70433.
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1631-073X/$ – see front matter 
doi:10.1016/j.crma.2005.12.006

*isomorphism. The universal deformation ring $R(S_4, V)$ of $V$ is isomorphic to $W[[t]]/(t^2, 2t)$. In particular, $R(S_4, V)$ is not a complete intersection ring.*

For background on deformation theory, see [4] and [6]. As in [5, §19.3], a commutative local Noetherian ring $R$ is a complete intersection ring if there is a regular complete local commutative Noetherian ring $S$ and a regular sequence $x_1, \ldots, x_n \in S$ such that the completion $\widehat{R}$ of $R$ with respect to powers of the maximal ideal is isomorphic to $S/(x_1, \ldots, x_n)$.

The fact that $W[[t]]/(t^2, 2t)$ is not a complete intersection ring is a consequence of [5, Proposition (19.3.2)], since the ideal $(t^2, 2t)$ cannot be generated by a regular sequence of elements in the regular local ring $W[[t]]$. To show $R(S_4, V) \cong W[[t]]/(t^2, 2t)$, which is a correction of [1, Proposition 4.2], we need the following lemma, which is a correction of [1, Lemma 4.1]. The proof is elementary, so we will omit it.

**Lemma 2.2.** *Let $W$ be the ring of infinite Witt vectors over $k$. Let $R$ be a complete local Noetherian $W$-algebra for which there is exactly one continuous surjection $\tau : R \to W$ of $W$-algebras and an isomorphism $\mu : R/2R \to k[s]/(s^2)$ of $W$-algebras. Then $R$ is isomorphic to $W[[t]]/(t^2 - 2\gamma t, \alpha 2^m t)$, where $\gamma \in W$, $\alpha \in \{0, 1\}$, $0 < m \in \mathbb{Z}$ and either $\gamma = 0$ or $\alpha = 1$.*

To prove Theorem 2.1, we notice that by [1, Proof of Proposition 4.2], there is exactly one continuous surjective $W$-algebra homomorphism $R = R(S_4, V) \to W$, and $R/2R \cong k[t]/(t^2)$. By Lemma 2.2, $R$ is isomorphic to $W[[t]]/(t^2 - 2\gamma t, \alpha 2^m t)$ for some $\gamma$ and $\alpha$ as in the lemma. Let $G = \langle u, v, r, s \mid \mathrm{Rel}\rangle$ with $\mathrm{Rel} = \{u^2 = v^2 = r^3 = s^2 = 1, uv = vu, srs = r^{-1}, sus = v, svs = u, rur^{-1} = v, rvr^{-1} = uv\}$. By letting $u = (1, 2)(3, 4)$, $v = (1, 4)(2, 3)$, $r = (1, 2, 3)$ and $s = (1, 3)$, we see that $G$ is isomorphic to $S_4$. We now construct a representation $\tau : G = S_4 \to \mathrm{GL}_2(W[[t]]/(t^2, 2t))$ which mod 2 gives a universal mod 2 deformation of $V$. Define $\tau$ by the following matrices:

$$\tau(u) = \begin{pmatrix} 1+t & t \\ 0 & 1+t \end{pmatrix}, \quad \tau(v) = \begin{pmatrix} 1+t & 0 \\ t & 1+t \end{pmatrix}, \quad \tau(r) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \tau(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The reduction $\bar\tau$ of $\tau$ mod 2 defines an indecomposable $kS_4$-module $\overline{U}$ satisfying $t\overline{U} \cong V$ and $\overline{U}/t\overline{U} \cong V$. It follows from $R(S_4, V)/2R(S_4, V) \cong k[t]/(t^2)$ that $\overline{U}$ is isomorphic to the universal mod 2 deformation of $V$ as $kS_4$-module. The reduction of $\tau$ mod $(t)$ defines a deformation of $V$ over $W$ and corresponds to the unique surjection $R(S_4, V) \to W = R(S_3, V)$ mentioned earlier.

Suppose now that $R$ is not isomorphic to $W[[t]]/(t^2, 2t)$ so that if $\alpha = 1$ then $m \geqslant 2$. Recall $\gamma = 0$ or $\alpha = 1$. To obtain a contradiction, we need to show there are no $\gamma$ and $\alpha$ as above such that $\tau$ can be lifted to $W[[z]]/(z^2 - 2\gamma z, \alpha 2^m z)$ for a continuous $W$-algebra homomorphism

$$\nu : R = W[[z]]/\bigl(z^2 - 2\gamma z, \alpha 2^m z\bigr) \to W[[t]]/\bigl(t^2, 2t\bigr)$$

which induces an isomorphism $R/2R \to k[t]/(t^2)$. One checks that $\nu(z) = \kappa t$ for some $\kappa \in W^*$, so on replacing $\gamma$ by $\kappa^{-1}\gamma$ we can reduce to the case in which $\nu(z) = t$. Since $W[[t]]/(t^2 - 2\gamma t, 4t)$ is a quotient algebra of $W[[z]]/(z^2 - 2\gamma z, \alpha 2^m z)$ through which $\nu$ factors, it is enough to show that $\tau$ cannot be lifted to $W[[t]]/(t^2 - 2\gamma t, 4t)$ for any $\gamma \in W$ for the canonical projection $\pi_\gamma : W[[t]]/(t^2 - 2\gamma t, 4t) \to W[[t]]/(t^2, 2t)$ sending $t$ to $t$.

This can be seen by looking at $\tau(u)$. If $\hat\tau$ were a lift of $\tau$ to $W[[t]]/(t^2 - 2\gamma t, 4t)$ for $\pi_\gamma$, then $\hat\tau(u)$ would be conjugate to a matrix $A_u$ over $W[[t]]/(t^2 - 2\gamma t, 4t)$ which has to satisfy the relation $A_u^2 \equiv I \bmod (t^2 - 2\gamma t, 4t)$, where $I$ denotes the identity $2 \times 2$ matrix. An easy matrix calculation shows that this is not possible. Hence $\tau$ cannot be lifted to $W[[t]]/(t^2 - 2\gamma t, 4t)$ for any $\gamma \in W$, which implies that $R = R(S_4, V) \cong W[[t]]/(t^2, 2t)$.

## 3. Capping groups

**Definition 3.1.** Let $\ell$ be a prime number, and suppose there is a short exact sequence

$$1 \to K \to \Gamma \xrightarrow{\pi} G \to 1$$

where $\Gamma$ and $G$ are profinite groups, $\pi$ is a continuous group homomorphism and $K$ is a closed normal subgroup of $\Gamma$. We say *$G$ caps $\Gamma$ (via $\pi$) for $\ell$* if there is no closed normal subgroup $K_0$ of $\Gamma$ satisfying $K_0 \subset K$ and for which $K/K_0$ is a non-trivial pro-$\ell$ group. By considering intersections of conjugate subgroups, this is equivalent to saying that there is no closed normal subgroup $K'$ of $K$ such that $K/K'$ is a non-trivial pro-$\ell$ group.

We now relate this concept to deformation theory. For simplicity, we suppose that $\Gamma$ satisfies the following $\ell$-finiteness condition of Mazur [6]. If $k$ is a perfect field of characteristic $\ell$, then $H^1(\Gamma, X)$ is finite dimensional over $k$ for all discrete finite dimensional representations $X$ over $k$. By [6], this implies that if $V$ is a discrete finite dimensional representation of $\Gamma$ over $k$, the versal deformation ring $R(\Gamma, V)$ is well defined. If $\mathrm{End}_{k\Gamma}(V) = k$ then $R(\Gamma, V)$ is a universal deformation ring.

Suppose now that $V$ is the inflation to $\Gamma$ of a representation of $G$ which we will also denote by $V$. Let $U(\Gamma, V)$ be a versal deformation of $V$ as a representation of $\Gamma$. The following result is a consequence of Lemma 2.1, Proposition 3.1 and Theorem 3.2 of [2].

**Lemma 3.2.** *Fix a perfect field $k$ of characteristic $\ell$. Let $M(\Gamma, G, k)$ be the set of discrete finite dimensional representations $V$ of $\Gamma$ over $k$ which are inflated from representations of $G$. If $\Gamma$ satisfies Mazur's $\ell$-finiteness condition, the following are equivalent*:

(a) *The group $G$ caps $\Gamma$ via $\pi$ for $\ell$.*
(b) *The group $K = \mathrm{Ker}(\pi : \Gamma \to G)$ acts trivially on $U(\Gamma, V)$ for all $V \in M(\Gamma, G, k)$.*

*In particular, if $G$ caps $\Gamma$ via $\pi$ for $\ell$ and $V \in M(\Gamma, G, k)$, then $R(\Gamma, V)$ is isomorphic to the versal deformation ring $R(G, V)$ of $V$ as a representation of $G$.*

**Definition 3.3.** Let $\ell$ be a prime, let $G$ be a profinite group, and let $L$ be a number field. Let $S$ be a finite set of places of $L$. Define $G_{L,S}$ to be the Galois group over $L$ of the maximal algebraic extension of $L$ which is unramified outside $S$.

(a) We say *$G$ caps $L$ for $\ell$ at $S$* if $G$ caps $G_{L,S}$ for $\ell$ via some surjection $G_{L,S} \to G$.
(b) We say *$G$ caps $L$ for $\ell$* if there is a set of places $S$ such that $G$ caps $G_{L,S}$ for $\ell$.
(c) We say *$G$ is a capping group* for $\ell$ if $G$ caps some number field $L$ for $\ell$.

The natural question in this context is:

**Question 1.** Given a prime $\ell$, which profinite groups $G$ are capping groups for $\ell$? Which of these cap $\mathbb{Q}$ for $\ell$?

One can phrase various statements in Iwasawa theory in terms of capping groups. For example, it follows from [9, Proposition 10.13] that an odd prime $\ell$ is regular, in the sense that $\ell$ does not divide the class number of the cyclotomic field $\mathbb{Q}(\zeta_\ell)$, if and only if the group $G = \mathbb{Z}_\ell^*/\{\pm 1\}$ caps $\mathbb{Q}$ for $\ell$ at $S = \{\ell\}$.

Our main result is:

**Theorem 3.4.** *Let $G$ be the symmetric group $S_4$.*

(a) *The group $G$ does not cap $\mathbb{Q}$ for $\ell = 2$.*
(b) *There are infinitely many real quadratic fields $L$ such that $G$ caps $L$ for $\ell = 2$ at the empty set $S$ of places of $L$.*

Part (b) in this theorem is equivalent to the statement that there are infinitely many real quadratic fields $L$ for which there is an unramified $S_4$-extension of $L$ which has odd class number. An example of such a field is $L = \mathbb{Q}(\sqrt{5 \cdot 14197})$.

**Corollary 3.5.** *Let $k$ be a perfect field of characteristic 2, let $V$ be a non-trivial irreducible $kS_4$-module of dimension 2. There are infinitely many real quadratic fields $L$ such that*

(a) *there is a surjection $\pi : G_{L,\emptyset} \to S_4$, and*
(b) *when $V$ is viewed as a module for $G_{L,\emptyset}$ via $\pi$, the ring $R(G_{L,\emptyset}, V) = R(S_4, V) \cong W[[t]]/(t^2, 2t)$ is not a complete intersection.*

The proofs of these results use a Theorem of Tate [8, Theorem 4] which states that every projective representation $\tilde{\rho}: G_F \to \mathrm{PGL}_2(\mathbb{C})$ of the absolute Galois group $G_F$ of a local or global field $F$ has a lifting to a linear representation $\rho: G_F \to \mathrm{GL}_2(\mathbb{C})$. Different liftings differ by tensoring by a one-dimensional character $\chi: G_F \to \mathbb{C}^*$. Part (a) of Theorem 3.4 results from composing a surjection $G_{\mathbb{Q},S} \to G = S_4$ for some set of places $S$ of $\mathbb{Q}$ with an injection of $S_4$ into $\mathrm{PGL}_2(\mathbb{C})$, leading to a projective representation $\tilde{\rho}$ as above which is unramified outside $S$. Tate proves that one can always construct a lifting $\rho$ of $\tilde{\rho}$ which is unramified outside $S \cup \{\infty\}$. For a suitable character $\chi: G_{\mathbb{Q}} \to \mathbb{C}^*$, the kernel of $\chi \otimes \rho$ defines a subgroup $K$ of $G_{\mathbb{Q},S}$ of the kind needed to show that $G = S_4$ does not cap $G_{\mathbb{Q},S}$ for $\ell = 2$.

To prove part (b) of Theorem 3.4, one starts by letting $F = \mathbb{Q}$ and $S = \{p\}$ for a prime $p \equiv 5 \bmod 8$ in the proof of part (a). After replacing $\rho$ by $\chi \otimes \rho$, this leads to representations $\tilde{\rho}$ and $\rho$ of $G_{\mathbb{Q},\{p\}}$. Let $N$ be the $S_4$-extension $\overline{\mathbb{Q}}^{\mathrm{kernel}(\tilde{\rho})}$ of $\mathbb{Q}$. One can find such $p$ and $\tilde{\rho}$ for which $p$ is quadratically ramified in $N$. Let $r \neq p$ be an auxiliary odd prime. Then $N(\sqrt{pr})$ is an unramified $S_4$-extension of $\mathbb{Q}(\sqrt{pr})$, and $\mathrm{Gal}(N(\sqrt{pr})/\mathbb{Q}(\sqrt{pr})) = S_4$ caps $G_{\mathbb{Q}(\sqrt{pr}),\emptyset}$ provided that the class number $h_{N(\sqrt{pr})}$ of $N(\sqrt{pr})$ is odd.

To show $h_{N(\sqrt{pr})}$ is odd for a suitable $p$ and $r$, one can reduce to showing that there is no Galois extension $E$ of $\mathbb{Q}$ containing $N(\sqrt{pr})$ which is unramified over $N(\sqrt{pr})$ and for which $T = \mathrm{Gal}(E/N(\sqrt{pr}))$ is a simple module for the group ring $(\mathbb{Z}/2)\,\mathrm{Gal}(N(\sqrt{pr})/\mathbb{Q})$. If such an $E$ exists, $\mathrm{Gal}(E/\mathbb{Q}(\sqrt{pr}))$ is an extension of $\mathrm{Gal}(N(\sqrt{pr})/\mathbb{Q}(\sqrt{pr})) = S_4$ by $T$, and one can analyze the possible groups which can occur. One then puts conditions on $r$ which preclude the existence of such an $E$. A first step in doing this is to analyze the possible liftings to $\mathrm{GL}_2(\mathbb{C})$ of a projective representation $G_{\mathbb{Q}(\sqrt{pr})} \to \mathrm{PGL}_2(\mathbb{C})$ which has kernel $\mathrm{Gal}(\overline{\mathbb{Q}}/N(\sqrt{pr}))$. Here one uses the fact that the restriction of $\rho$ to $G_{\mathbb{Q}(\sqrt{pr})}$ is one such lifting, and all liftings differ by twists by a one-dimensional character of $G_{\mathbb{Q}(\sqrt{pr})}$. Having found one $p$ and $r$ for which $h_{N(\sqrt{pr})}$ is odd (e.g., $(p, r) = (14197, 5)$ according to Pari [7]) one can show that there is a positive Dirichlet density of primes $q$ such that $h_{N(\sqrt{pq})}$ is odd. One can take such a set of $q$ to consist of those odd primes different from $p$ which have the same Frobenius conjugacy class as $r$ in $\mathrm{Gal}(N'/\mathbb{Q})$, where $N'$ is the finite extension of $N$ which results from adjoining the square roots of every unit of $N$.

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