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Partial Differential Equations/Functional Analysis

Convex Sobolev inequalities and spectral gap

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Abstract

This Note is devoted to the proof of convex Sobolev (or generalized Poincaré) inequalities which interpolate between spectral gap (or Poincaré) inequalities and logarithmic Sobolev inequalities. We extend to the whole family of convex Sobolev inequalities results which have recently been obtained by Cattiaux, and Carlen and Loss for logarithmic Sobolev inequalities. Under local conditions on the density of the measure with respect to a reference measure, we prove that spectral gap inequalities imply all convex Sobolev inequalities including in the limit case corresponding to the logarithmic Sobolev inequalities. ***To cite this article: J.-P. Bartier, J. Dolbeault, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Inégalités de Sobolev convexes et trou spectral. Cette Note est consacrée à la preuve d'inégalités de Sobolev convexes (ou inégalités de Poincaré généralisées) qui interpolent entre des inégalités de trou spectral (ou de Poincaré) et des inégalités de Sobolev logarithmiques. Nous étendons à la famille des inégalités de Sobolev convexes toute entière des résultats qui ont été obtenus récemment par Cattiaux, et Carlen et Loss pour des inégalités de Sobolev logarithmiques. Sous des conditions locales sur la densité de la mesure par rapport à une mesure de référence, nous démontrons que les inégalités de trou spectral entraînent toutes les inégalités de Sobolev convexes, y compris dans le cas limite des inégalités de Sobolev logarithmiques. ***Pour citer cet article : J.-P. Bartier, J. Dolbeault, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Soit μ une mesure de probabilité sur $\mathbb{R}^d$. On dira que μ admet une *inegalité de Sobolev logarithmique* (*tendue*) s'il existe une constante $C_1(\mu)$ telle que

$$\int u^2 \log\left(\frac{u^2}{\|u\|^2_{L^2(\mu)}}\right) \mathrm{d}\mu \leqslant C_1(\mu) \|\nabla u\|^2_{L^2(\mu)} \quad \forall u \in H^1(\mu), \tag{1}$$

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et une *inégalité de Poincaré*, ou encore de *trou spectral*, s'il existe une constante $C_2(\mu)$ telle que

$$\int |u-\bar{u}|^2 \, \mathrm{d}\mu \leqslant C_2(\mu) \|\nabla u\|^2_{L^2(\mu)} \quad \text{avec } \bar{u} := \int u \, \mathrm{d}\mu, \ \forall u \in H^1(\mu). \tag{2}$$

Si l'inégalité (1) est vérifiée, alors (2) est aussi vraie avec $C_2(\mu) \leqslant \frac{1}{2} C_1(\mu)$. La réciproque est fausse en général (considérer $\mathrm{d}\mu(x) = C \exp(-|x|^\alpha)$ avec $\alpha \in [1,2)$, voir [8,11,14,6,5]). Cependant, Cattiaux dans [10], Wang dans [16] puis Carlen et Loss dans [9] ont donné des conditions suffisantes sur μ pour que (1) se déduise de (2). Le but de cette note est d'améliorer certains de ces résultats en considérant une famille d'inégalités qui interpole entre (1) et (2). A la suite de Beckner [7], pour $p \in (1,2]$, on dira que μ vérifie une *inégalité de Poincaré généralisée* s'il existe une constante positive finie $C_p(\mu)$ telle que

$$\frac{1}{p-1}\left[\int |u|^2 \, \mathrm{d}\mu - \left(\int |u|^{2/p} \, \mathrm{d}\mu\right)^p\right] \leqslant C_p(\mu) \int |\nabla u|^2 \, \mathrm{d}\mu \quad \forall u \in H^1(\mu). \tag{3}$$

Le cas $p=2$ correspond à (2) et dans la limite $p \to 1$, on retrouve (1) si $\liminf_{p\to 1} C_p(\mu)$ est finie. On peut montrer que

$$\frac{2}{p} C_2(\mu) \leqslant C_p(\mu) \quad \forall p \in [1,2] \quad \text{et} \quad C_p(\mu) \leqslant \min\left(\frac{1}{p-1} C_2(\mu), \, pC_1\right) \quad \forall p \in (1,2] \tag{4}$$

(voir [1,14]). Il n'est pas possible d'en déduire une estimation de $C_1(\mu)$ sachant seulement que $C_2(\mu)$ est finie. Dans la suite, $C_p(\mu)$ désignera pour tout $p \in [1,2]$ la valeur optimale de la constante.

Gross a montré dans [12] que (1) est vérifiée pour les mesures gaussiennes, $\nu_\sigma(x) := (2\pi\sigma^2)^{-d/2} \, \mathrm{e}^{-|x|^2/(2\sigma^2)}$, et, en utilisant les polynômes d'Hermite, Beckner a établi dans [7] que (3) est aussi vérifiée lorsque $\mu = \nu_\sigma$, pour tout $p \in (1,2)$, avec $C_p(\nu_\sigma) = 2\sigma^2/p$. La méthode d'entropie – production d'entropie de Bakry et Émery [4] permet de montrer (3) dans le cas de mesures du type $\mu = \mathrm{e}^{-V}$ lorsque V est strictement convexe, voir [2]. Pour cette raison, les inégalités (3) sont aussi appelées *inégalités de Sobolev convexes*. On montre ainsi que $C_p(\mathrm{e}^{-V})$ peut être borné en fonction de $\inf_{\xi \in S^{d-1}, \, x \in \mathbb{R}^d} (D^2 V(x)\xi, \xi)$. Cela conduit naturellement à rechercher des conditions suffisantes sur V pour borner $C_1(\mu)$ en fonction de $C_2(\mu)$, ou, en d'autres termes, pour que l'inégalité de Poincaré entraîne l'inégalité de Sobolev logarithmique. Nous allons plus généralement nous intéresser à des estimations de $C_p(\mu)$ pour tout $p \in (1,2)$. En prenant la limite $p \to 1$, on retrouve et on améliore les résultats obtenus pour $p=1$ dans [10,9]. Notre principal résultat est un résultat de perturbation pour les inégalités de Sobolev convexes (3) qui diffère toutefois de la méthode classique de Holley–Stroock [13,2,1].

Théorème 0.1. *Soit $p \in [1,2)$ et $p' = (1-1/p)^{-1}$ si $p > 1$, $p' = \infty$ si $p = 1$. Si μ et ν sont deux mesures de probabilités de densités respectives e^{-V} et e^{-W} par rapport à la mesure de Lebesgue telles que $C_p(\nu)$ et $C_2(\mu)$ soient finies, si*

$$m := \inf_{x \in \mathbb{R}^d} \left(|\nabla Z|^2 - \Delta Z + \nabla Z \cdot \nabla W\right) > -\infty,$$

avec $Z := \frac{1}{2}(V - W)$, et si $Z_+ \in L^{p'}(\nu)$, alors

$$C_p(\mu) \leqslant \mathcal{C}_p := \frac{2}{p} C_2(\mu) + \left(\frac{2}{p} - 1\right) \mathcal{C}_p^* \quad \text{avec } \mathcal{C}_p^* := \left[C_p(\nu) + C_2(\mu)\left(2\|Z_+\|_{L^{p'}(\nu)} - m C_p(\nu)\right)_+\right].$$

La preuve du Théorème 0.1 consiste comme dans [9] pour $p=1$ à établir d'abord une inégalité restreinte :

$$\frac{\int |v|^2 \, \mathrm{d}\mu - (\int |v|^{2/p} \, \mathrm{d}\mu)^p}{(p-1) \int |\nabla v|^2 \, \mathrm{d}\mu} \leqslant \mathcal{C}_p^* \quad \forall v \in H^1(\mu) \text{ tel que } \bar{v} = 0,$$

puis, grâce à l'inégalité $(\int |u|^q \, \mathrm{d}\mu)^{2/q} \geqslant |\bar{u}|^2 + (q-1)(\int |u - \bar{u}|^q \, \mathrm{d}\mu)^{2/q}$ pour tout $q \in [1,2]$ et pour tout $u \in L^1 \cap L^q(\mu)$ (cf. [6]), on montre le résultat avec $\mathcal{C}_p := \frac{2}{p} C_2(\mu) + (\frac{2}{p} - 1)\mathcal{C}_p^*$.

1. Introduction and main result

Consider a probability measure μ on $\mathbb{R}^d$. We say that there is a (*tight*) *logarithmic Sobolev inequality* associated to μ if there exists a finite constant $C_1(\mu)$ such that

$$\int u^2 \log\left(\frac{u^2}{\|u\|^2_{L^2(\mu)}}\right) d\mu \leqslant C_1(\mu)\|\nabla u\|^2_{L^2(\mu)} \quad \forall u \in H^1(\mu), \tag{1}$$

and a *spectral gap* or *Poincaré inequality* associated to μ if there exists a finite constant $C_2(\mu)$ such that

$$\int |u-\bar{u}|^2\, d\mu \leqslant C_2(\mu)\|\nabla u\|^2_{L^2(\mu)} \quad \text{with } \bar{u} := \int u\, d\mu,\ \forall u \in H^1(\mu). \tag{2}$$

It is well known that if (1) holds, then (2) is also true with $C_2(\mu) \leqslant \frac{1}{2}C_1(\mu)$. This is easily checked by writing $u = 1+\varepsilon v$, with $\bar{v} = 0$, and by letting $\varepsilon \to 0$. The reverse implication is a much harder question, which is sometimes false. An example of such a situation is given by $\mu(x) = \exp(-|x|^\alpha)$ in $\mathbb{R}^d$ with $\alpha \in [1,2)$, see, e.g., [8,11] and [14,6,5] for more details. Cattiaux in [10], Wang in [16] and then Carlen and Loss in [9] with more elementary tools, gave sufficient conditions on μ under which (1) is a consequence of (2). The goal of this note is to revisit some of these results by considering a family of inequalities which interpolate between (1) and (2).

According to Beckner in [7], we shall say that, for some $p \in (1,2]$, there is a *generalized Poincaré inequality* associated to μ if there exists a positive constant $C_p(\mu)$ such that

$$\frac{1}{p-1}\left[\int |u|^2\, d\mu - \left(\int |u|^{2/p}\, d\mu\right)^p\right] \leqslant C_p(\mu)\int |\nabla u|^2\, d\mu \quad \forall u \in H^1(\mu). \tag{3}$$

Throughout this paper, we will assume that for any $p \in [1,2]$, $C_p(\mu)$ is the optimal constant. We will not consider "defective" logarithmic Sobolev inequality (see, e.g., [10,15]) and will omit the word "tight" whenever we mention inequality (1). The limit case $p = 2$ of (3) corresponds to (2) for nonnegative solutions, but (2) and (3) look different otherwise. In that case we indeed get an inequality which involves the average of $|u|$ instead of the average of u. However, if (3) with $p = 2$ holds for a function $u - a = v \geqslant 0$, a straightforward computation shows that (2) also holds for u: (2) holds for any $u \in H^1(\mu)$ such that $u_- \in L^\infty(\mu)$. By density, we extend it to any $u \in H^1(\mu)$: (3) with $p = 2$ is therefore equivalent to (2). On the other hand, by taking the limit $p \to 1$ in (3), we find $C_1(\mu) \leqslant \liminf_{p\to 1} C_p(\mu)$, which proves (1) if the right-hand side is finite. By considering again $u = 1+\varepsilon v$, with $\bar{v} = 0$, in the limit $\varepsilon \to 0$, we get:

$$C_p(\mu) \geqslant \frac{2}{p} C_2(\mu) \quad \forall p \in [1,2].$$

By Hölder's inequality, $(\int u\, d\mu)^2 \leqslant (\int |u|^{2/p}\, d\mu)^p$ for any $p \in [1,2]$. As a consequence, for any $p \in (1,2]$,

$$C_p(\mu) = \sup_{u \in H^1(\mu)} \frac{\int |u|^2\, d\mu - (\int |u|^{2/p}\, d\mu)^p}{(p-1)\int |\nabla u|^2\, d\mu} \leqslant \frac{1}{p-1}\sup_{u \in H^1(\mu)} \frac{\int |u|^2\, d\mu - (\int u\, d\mu)^2}{\int |\nabla u|^2\, d\mu} = \frac{C_2(\mu)}{p-1}. \tag{4}$$

We refer to [1] for more details. Combining these two inequalities, we get: $C_p(\mu) \leqslant \frac{1}{2}C_1(\mu)/(p-1)$. In [14], an improved estimate is established: $C_p \leqslant pC_1$ (also see [1]). However, at this stage, it is clear that we have no estimate on $C_1(\mu)$ if we only know that $C_p(\mu)$ is finite for some $p \in (1,2]$.

Inequality (1) has been established by Gross in [12] in the case of Gaussian measures, $\nu_\sigma(x) := (2\pi\sigma^2)^{-d/2} \times \exp[-|x|^2/(2\sigma^2)]$, and using Hermite polynomials Beckner in [7] proved that (3) holds with $C_p(\nu_\sigma) = 2\sigma^2/p$. An alternative method based on the entropy – entropy production method of Bakry and Emery [4] has been adapted in [2] to prove (3) in more general situations which for instance cover the case of measures $\mu = e^{-V}$ for some strictly convex function V. For this reason, the family of inequalities (3) has been called *convex Sobolev inequalities*. The entropy – entropy production method gives an upper bound on $C_p(e^{-V})$ involving $\inf_{\xi \in S^{d-1},\, x \in \mathbb{R}^d}(D^2V(x)\xi,\xi)$. This shows that at least in some circumstances, the bounds in (4) are not optimal. Our purpose is to give sufficient conditions on V to bound $C_p(\mu)$ for any $p \in (1,2)$ in terms of $C_2(\mu)$. We also recover and improve the results of [10,16,9] for $p = 1$ by deriving uniform estimates in the limit $p \to 1$.

Note that the strict convexity condition on V can be relaxed using Holley–Stroock type estimates [13,2,1] and L^∞ perturbations of V can be considered. Although it differs in nature from the Holley and Stroock perturbation lemma, our main result can be seen as a perturbation result as well.

Theorem 1.1. *Let $p \in [1,2)$ and $p' = (1-1/p)^{-1}$ if $p > 1$, $p' = \infty$ if $p = 1$. Let μ and ν be two probability measures with respective densities e^{-V} and e^{-W} relatively to Lebesgue's measure such that $C_p(\nu)$ and $C_2(\mu)$ are finite. Let $Z := \frac{1}{2}(V - W)$ and assume that $Z_+ \in L^{p'}(\nu)$, $m := \inf_{\mathbb{R}^d}(|\nabla Z|^2 - \Delta Z + \nabla Z \cdot \nabla W) > -\infty$. Then we have*

$$C_p(\mu) \leqslant \mathcal{C}_p := \frac{2}{p} C_2(\mu) + \left(\frac{2}{p} - 1\right)\mathcal{C}_p^*, \quad \text{with } \mathcal{C}_p^* := \left[C_p(\nu) + C_2(\mu)\left(2\|Z_+\|_{L^{p'}(\nu)} - m C_p(\nu)\right)_+\right].$$

Here, m_+ is the positive part of m: $m_+ := \max\{0, m\}$. By relative density, we simply mean, e.g., $\mathrm{d}\mu(x) = \mathrm{e}^{-V(x)}\,\mathrm{d}x$. With these notations, $\mu = \mathrm{e}^{-2Z}\nu$. Notice that the limit case $p = 1$, which corresponds to the logarithmic Sobolev inequalities (1), is covered. To compare our results with those of [9], we can state a result in the Euclidean space for generalized Poincaré inequalities corresponding to Gaussian weights, i.e. $\nu = \nu_\sigma$, $W(x) = |x|^2/(2\sigma^2)$, and recover in the limit $p \to 1$ the logarithmic Sobolev inequality. The freedom in the choice of the parameter σ corresponds to the scaling invariance in the Euclidean space. For $\nu = \nu_\sigma$, inequality (3) holds. Theorem 1.1 becomes

Corollary 1.2. *Let $\nu = \nu_\sigma := (2\pi\sigma^2)^{-d/2}\exp[-|x|^2/(2\sigma^2)]$. If* (2) *holds for $\mu = \mathrm{e}^{-V}$, with V such that $\int \mathrm{e}^{-V}\,\mathrm{d}x = 1$, and if for some $\sigma > 0$, $\inf_{x\in\mathbb{R}^d}(|\nabla V(x)|^2 - 2\Delta V(x) - |x|^2/\sigma^4) > -\infty$, then inequalities* (1) *for $p = 1$ and* (3) *for any $p \in (1,2)$ also hold, provided $V - |x|^2/(2\sigma^2)$ is bounded from above in the first case and*

$$\int_{\mathbb{R}^d} \left(V - \frac{|x|^2}{2\sigma^2}\right)_+^{p'} \exp\left[-\frac{|x|^2}{2\sigma^2}\right]$$

is finite in the second case.

2. Proof of the main result

As in [9], we first prove Theorem 1.1 in the *restricted case* which corresponds to $\bar{u} = 0$ and then extend it to the *unrestricted case.*

Lemma 2.1. *Under the assumptions of Theorem* 1.1*,*

$$\sup_{v\in H^1(\mu),\ \bar{v}=0} \frac{\int |v|^2\,\mathrm{d}\mu - (\int |v|^{2/p}\,\mathrm{d}\mu)^p}{(p-1)\int |\nabla v|^2\,\mathrm{d}\mu} \leqslant \mathcal{C}_p^*.$$

Proof. Assume first that $p > 1$. We look for some $t > 0$ such that for any v in $H^1(\mu)$ with $\bar{v} = 0$,

$$\mathcal{A}(t) := \|\nabla v\|^2_{L^2(\mu)} - \frac{t}{(p-1)C_p(\nu)}\left[\int |v|^2\,\mathrm{d}\mu - \left(\int |v|^{2/p}\,\mathrm{d}\mu\right)^p\right]$$

is nonnegative. Let $\mathcal{A}(t) = \text{(I)} + \text{(II)} + \text{(III)}$ with

$$\text{(I)} = (1-t)\int |\nabla v|^2\,\mathrm{d}\mu, \quad \text{(II)} = t\int |\nabla v|^2\,\mathrm{d}\mu \quad \text{and}$$

$$\text{(III)} = \frac{-t}{(p-1)C_p(\nu)}\left[\int |v|^2\,\mathrm{d}\mu - \left(\int |v|^{2/p}\,\mathrm{d}\mu\right)^p\right].$$

Define g such that $v = g\mathrm{e}^Z$: $\int |v|^2\,\mathrm{d}\mu = \int |g|^2\,\mathrm{d}\nu$ and $\int |\nabla v|^2\,\mathrm{d}\mu = \int |\nabla g|^2\,\mathrm{d}\nu + \int \delta |g|^2\,\mathrm{d}\nu$, where $\delta := |\nabla Z|^2 - \Delta Z + \nabla Z \cdot \nabla W$. Using the spectral gap assumption on μ, we get

$$\text{(I)} \geqslant \frac{1-t}{C_2(\mu)}\int |v|^2\,\mathrm{d}\mu = \frac{1-t}{C_2(\mu)}\int |g|^2\,\mathrm{d}\nu.$$

Using the fact that (3) holds for ν and the above expression of $\int |\nabla v|^2\,\mathrm{d}\mu$, we obtain

$$\text{(II)} \geqslant \frac{t}{(p-1)C_p(\nu)}\left(\int |g|^2\,\mathrm{d}\nu - \left(\int |g|^{2/p}\,\mathrm{d}\nu\right)^p\right) + t\int \delta|g|^2\,\mathrm{d}\nu.$$

As for the last term, we can write it as

$$\text{(III)} = \frac{t}{(p-1)C_p(\nu)}\left(\left(\int |v|^{2/p}\,\mathrm{d}\mu\right)^p - \int |g|^2\,\mathrm{d}\nu\right).$$

Collecting these estimates, we have

$$\mathcal{A}(t) \geqslant \int\left(\frac{(1-t)}{C_2(\mu)} + t\delta\right)|g|^2\,\mathrm{d}\mu + \frac{\mathcal{B}t}{(p-1)C_p(\nu)}, \quad \mathcal{B} := \left(\int |v|^{2/p}\,\mathrm{d}\mu\right)^p - \left(\int |g|^{2/p}\,\mathrm{d}\nu\right)^p. \tag{5}$$

Let $\mathrm{d}\pi := |g|^{2/p}/\int |g|^{2/p}\,\mathrm{d}\nu$. By Jensen's inequality applied to the convex function $t \mapsto \mathrm{e}^{-t}$, we get

$$\frac{\int |v|^{2/p}\,\mathrm{d}\mu}{\int |g|^{2/p}\,\mathrm{d}\nu} = \frac{\int |g|^{2/p}\mathrm{e}^{-2(1-1/p)Z}\,\mathrm{d}\nu}{\int |g|^{2/p}\,\mathrm{d}\nu} = \int \mathrm{e}^{-2(1-1/p)Z}\,\mathrm{d}\pi \geqslant \exp\left[-2\left(1-\frac{1}{p}\right)\int Z\,\mathrm{d}\pi\right].$$

Using the lower estimate $\mathrm{e}^{-t} \geqslant 1-t$ and $-Z \geqslant -Z_+$, we infer that

$$\left(\int |v|^{2/p}\,\mathrm{d}\mu\right)^p - \left(\int |g|^{2/p}\,\mathrm{d}\nu\right)^p \geqslant -2(p-1)\int Z_+|g|^{2/p}\,\mathrm{d}\nu\left(\int |g|^{2/p}\,\mathrm{d}\nu\right)^{p-1}.$$

By Hölder's inequality, we have $\|g\|_{L^{2/p}(\nu)} \leqslant \|g\|_{L^2(\nu)}$ and $\int Z_+|g|^{2/p}\,\mathrm{d}\nu \leqslant \|g\|^{2/p}_{L^2(\nu)}\|Z_+\|_{L^{p'}(\nu)}$, which amounts to $\mathcal{B} \geqslant -2(p-1)\|Z_+\|_{L^{p'}(\mu)}\int |g|^2\,\mathrm{d}\nu$. Altogether, we get

$$\mathcal{A}(t) \geqslant \left[\frac{1-t}{C_2(\mu)} + t\left(m - \frac{2\|Z_+\|_{L^{p'}(\nu)}}{C_p(\nu)}\right)\right]\int |g|^2\,\mathrm{d}\nu.$$

This proves that $\mathcal{A}(t) \geqslant 0$ for any $t \in (0, t^*]$ with $t^* := [1 + \frac{C_2(\mu)}{C_p(\nu)}(2\|Z_+\|_{L^{p'}(\nu)} - mC_p(\nu))_+]^{-1}$ and proves the result with $\mathcal{C}_p = C_p(\nu)/t^*$.

The case $p = 1$ is easily obtained as follows. By the method of [14], we know that if $C_1(\nu)$ is finite, then $C_p(\nu)$ is also finite for any $p \in (1,2]$ and $\lim_{p\to 1} C_p(\nu) = C_1(\nu)$. Since $\|Z_+\|_{L^{p'}(\mu)} \leqslant \|Z_+\|_{L^\infty(\mu)}$, we can bound $C_p(\mu)$ uniformly and pass to the limit as $p \to 1$. Notice that the limit case $p = 1$, by adding and subtracting $\int |v|^2\,\mathrm{d}\mu = \int |g|^2\,\mathrm{d}\nu$, consistently with the above calculations, we obtain

$$\lim_{p\to 1}\frac{\mathcal{B}}{p-1} = \int |g|^2\log\left(\frac{|g|^2}{\|g\|^2_{L^2(\nu)}}\right)\mathrm{d}\nu - \int |v|^2\log\left(\frac{|v|^2}{\|v\|^2_{L^2(\nu)}}\right)\mathrm{d}\mu = -2\int Z|g|^2\,\mathrm{d}\nu. \qquad \square$$

The general case $\bar{u} \neq 0$ in Theorem 1.1 is a consequence of the following estimate, see [6,17], which is the counterpart for $p < 2$ of a lemma given in [3] for $p > 2$.

Lemma 2.2. [6] *Let $q \in [1,2]$. For any function $u \in L^1 \cap L^q(\mu)$, if $\bar{u} := \int u\,\mathrm{d}\mu$, then*

$$\left(\int |u|^q\,\mathrm{d}\mu\right)^{2/q} \geqslant |\bar{u}|^2 + (q-1)\left(\int |u-\bar{u}|^q\,\mathrm{d}\mu\right)^{2/q}.$$

We refer to [6] for a complete proof and simply sketch the main idea at a formal level: Let $v := u - \bar{u}$, $\phi(t) := (\int |\bar{u} + tv|^q\,\mathrm{d}\mu)^{2/q}$, so that $\phi(0) = |\bar{u}|^2$, $\phi'(0) = 0$, $\phi(1) = (\int |u|^q\,\mathrm{d}\mu)^{2/q}$ and $\frac{1}{2}\phi''(t) \geqslant (q-1)(\int |v|^q\,\mathrm{d}\mu)^{2/q}$. This proves that $\phi(1) \geqslant \phi(0) + (q-1)(\int |v|^q\,\mathrm{d}\mu)^{2/q}$.

Proof of Theorem 1.1. Let $v := u - \bar{u}$ and apply Lemma 2.2 with $q = 2/p \in [1,2)$. Since

$$\int |u|^2\,\mathrm{d}\mu - |\bar{u}|^2 = \int |u-\bar{u}|^2\,\mathrm{d}\mu = \int |v|^2\,\mathrm{d}\mu,$$

we can write

$$\int |u|^2 \, d\mu - \left(\int |u|^{2/p} \, d\mu \right)^p \leqslant 2 \frac{p-1}{p} \int |v|^2 \, d\mu + \frac{2-p}{p} \left[\int |v|^2 \, d\mu - \left(\int |v|^{2/p} \, d\mu \right)^p \right].$$

We can then apply (2) and Lemma 2.1, and the result holds with $\mathcal{C}_p = \frac{2}{p} C_2(\mu) + (\frac{2}{p} - 1)\mathcal{C}_p^*$. □

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