Available online at www.sciencedirect.com
SCIENCE DIRECT
C. R. Acad. Sci. Paris, Ser. I 342 (2006) 79–81
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Number Theory

On the nontrivial zeros of modified Epstein zeta functions ☆

Haseo Ki

Department of Mathematics, Yonsei University, Seoul 120–749, South Korea

Received 11 January 2005; accepted 16 November 2005

Presented by Alain Connes

Abstract

We study the zeros of modified Epstein zeta functions having functional equations. The result is that for any $\delta > 0$, all but finitely many nontrivial zeros of such a function in $\{s \in \mathbb{C}\colon |s - \frac{1}{2}| < \delta\}$ are simple and on the critical line. As an immediate consequence of this theorem, all but finitely many nontrivial zeros of many modified Epstein zeta functions are simple and on the critical line. ***To cite this article: H. Ki, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Sur les zéros non-triviaux des fonctions Zeta modifiées d'Epstein. Nous étudions les zéros des fonctions Zeta modifiées d'Epstein ayant des équations fonctionnelles. Notre résultat est, que pour tout $\delta > 0$, tous les zéros non-triviaux d'une telle fonction dans $\{s \in \mathbb{C}\colon |s - \frac{1}{2}| < \delta\}$ sauf au plus un nombre fini d'entre eux, sont simples et sur la droite critique. Une conséquence immédiate de ce théorème est que tous les zéros non-triviaux, sauf au plus un nombre fini d'entre eux, de beaucoup de fonctions Zeta modifiées d'Epstein sont simples et sur la droite critique. ***Pour citer cet article : H. Ki, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction

The Epstein zeta function is

$$Z_Q(s) = {\sum_{m,n}}' Q(m,n)^{-s} \quad (\mathrm{Re}(s) > 1),$$

where the summation runs over all integers m, n ($(m,n) \neq (0,0)$) and $Q(u,v) = au^2 + buv + cv^2$ is a positive definite quadratic form. The function $Z_Q(s)$ can be continued analytically to the whole complex plane, except for $s = 1$, and it satisfies

$$\left(\frac{\sqrt{\Delta}}{2\pi}\right)^s \Gamma(s) Z_Q(s) = \left(\frac{\sqrt{\Delta}}{2\pi}\right)^{1-s} \Gamma(1-s) Z_Q(1-s),$$

where $\Delta = 4ac - b^2 > 0$.

☆ This work was supported by grant No. R01-2005-000-10339-0 from the Basic Research Program of the Korea Science & Engineering Foundation.

E-mail address: haseo@yonsei.ac.kr (H. Ki).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2005.11.015

We recall the Chowla–Selberg formula [1]:

$$Z_Q(s) = 2\zeta(2s)a^{-s} + \frac{2^{2s}a^{s-1}\sqrt{\pi}}{\Gamma(s)\Delta^{s-1/2}}\zeta(2s-1)\Gamma\left(s-\frac{1}{2}\right) + \frac{4\pi^s 2^{s-1/2}}{\sqrt{a}\Gamma(s)\Delta^{s/2-1/4}}\sum_{n=1}^{\infty} n^{s-1/2}\sum_{d|n} d^{1-2s}\cos\frac{n\pi b}{a}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{\pi n\sqrt{\Delta}}{a}\cosh t}\,\mathrm{e}^{(s-1/2)t}\,\mathrm{d}t.$$

In [2], the author investigated the distribution of zeros of truncations of the Epstein zeta function using the Chowla–Selberg formula. In particular, the author showed that for any positive integer N, all but finitely many nontrivial zeros of

$$2\zeta(2s)a^{-s} + \frac{2^{2s}a^{s-1}\sqrt{\pi}}{\Gamma(s)\Delta^{s-1/2}}\zeta(2s-1)\Gamma\left(s-\frac{1}{2}\right) + \frac{4\pi^s 2^{s-1/2}}{\sqrt{a}\,\Gamma(s)\Delta^{s/2-1/4}}\sum_{n=1}^{N} n^{s-1/2}\sum_{d|n} d^{1-2s}\cos\frac{n\pi b}{a}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{\pi n\sqrt{\Delta}}{a}\cosh t}\,\mathrm{e}^{(s-1/2)t}\,\mathrm{d}t$$

are simple and on the line $\mathrm{Re}(s) = \frac{1}{2}$ if $\frac{\sqrt{\Delta}}{2a} \geqslant 1$.

In this Note, we introduce more results that can be proved by the methods in [2].

Let α and β be real polynomials such that $\beta(s) = \beta(1-s)$ for any $s \in \mathbb{C}$. We define $Z_{Q,\alpha,\beta}(s)$ as follows:

$$Z_{Q,\alpha,\beta}(s) = 2\alpha(s)\zeta(2s)a^{-s} + \alpha(1-s)\frac{2^{2s}a^{s-1}\sqrt{\pi}}{\Gamma(s)\Delta^{s-1/2}}\zeta(2s-1)\Gamma\left(s-\frac{1}{2}\right) + \beta(s)\frac{4\pi^s 2^{s-1/2}}{\sqrt{a}\Gamma(s)\Delta^{s/2-1/4}}\sum_{n=1}^{\infty} n^{s-1/2}\sum_{d|n} d^{1-2s}\cos\frac{n\pi b}{a}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{\pi n\sqrt{\Delta}}{a}\cosh t}\,\mathrm{e}^{(s-1/2)t}\,\mathrm{d}t.$$

We immediately get

$$Z_{Q,\alpha,\beta}(s) = 2\big(\alpha(s)-\beta(s)\big)\zeta(2s)a^{-s} + \big(\alpha(1-s)-\beta(s)\big)\frac{2^{2s}a^{s-1}\sqrt{\pi}}{\Gamma(s)\Delta^{s-1/2}}\zeta(2s-1)\Gamma\left(s-\frac{1}{2}\right) + \beta(s)Z_Q(s).$$

Thus we obtain

$$Z_{Q,\alpha,\beta}(s) = Z_Q(s)$$

for $\alpha(s) = \beta(s) = 1$. It is easy to see that

$$\left(\frac{\sqrt{\Delta}}{2\pi}\right)^s \Gamma(s)Z_{Q,\alpha,\beta}(s) = \left(\frac{\sqrt{\Delta}}{2\pi}\right)^{1-s}\Gamma(1-s)Z_{Q,\alpha,\beta}(1-s).$$

We shall deal with more general functions related to $Z_{Q,\alpha,\beta}(s)$. For this purpose, we briefly introduce the following.

Let $L_1(s), \ldots, L_n(s)$ be Dirichlet series which can be analytically continued to the complex plane, except for finitely many points. We assume that

(i) $y > 0$;
(ii) α and β be real polynomials such that $\deg(\alpha) \geqslant \deg(\beta) + 1$ and $\beta(s) = \beta(1-s)$;
(iii) a_k is real for $k = 1, \ldots, n$;
(iv) $\pi^{-s}\Gamma(s)L_k(s) = \pi^{-1+s}\Gamma(1-s)L_k(1-s)$ $(k = 1, 2, \ldots, n,\ s \in \mathbb{C} \setminus \{0, 1\})$;
(v) for some $\epsilon > 0$, $L_k(s) = \mathrm{O}(|s|^{1-\epsilon})$ $(k = 1, 2, \ldots, n,\ \mathrm{Re}(s) \geqslant \frac{1}{2})$.

We define $Z(s; y, \alpha, \beta, L_1, \ldots, L_n)$ by

$$Z(s; y, \alpha, \beta, L_1, \ldots, L_n) = \alpha(s)\zeta(2s) + \alpha(1-s)\sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1)y^{1-2s} + y^{-s}\beta(s)\sum_{k=1}^{n} a_k L_k(s).$$

By a convexity argument, we can show that for any $\epsilon > 0$,

$$Z_Q(1/2 + \mathrm{i}t) = \mathrm{O}\big(|t|^{1/2+\epsilon}\big).$$

Thus we note that any $Z_{Q,\alpha,\beta}(s)$ is equal to some $Z(s; y, \alpha, \beta, L_1, \ldots, L_n)$. It is not hard to see that $Z(s; y, \alpha, \beta, L_1, \ldots, L_n)$ satisfies

$$\pi^{-s}\Gamma(s)Z(s; y, \alpha, \beta, L_1, \ldots, L_n) = \pi^{-1+s}\Gamma(1-s)Z(1-s; y, \alpha, \beta, L_1, \ldots, L_n).$$

We have the following theorem:

Theorem 1.1. *Let $\delta > 0$. Then all but finitely many zeros of $Z(s; y, \alpha, \beta, L_1, \ldots, L_n)$ in $\{s \in \mathbb{C}\colon |\mathrm{Re}(s) - \frac{1}{2}| < \delta\}$ are simple and on $\mathrm{Re}(s) = \frac{1}{2}$.*

Essentially, the proof of Theorem 1.1 follows from the methods in [2]. We have an immediate corollary from Theorem 1.1:

Corollary 1.2. *All but finitely many nontrivial zeros of $Z(s; y, \alpha, \beta, L_1, \ldots, L_n)$ are simple and on $\mathrm{Re}(s) = \frac{1}{2}$, provided that $y \geqslant 1$.*

We also obtain the following two corollaries from Theorem 1.1:

Corollary 1.3. *Let α and β be real polynomials such that $\deg(\alpha) \geqslant \deg(\beta) + 1$ and $\beta(s) = \beta(1-s)$ for any s in the complex plane. Then for any $\delta > 0$, all but finitely many zeros of $Z_{Q,\alpha,\beta}(s)$ in $\{s \in \mathbb{C}\colon |\mathrm{Re}(s) - \frac{1}{2}| < \delta\}$ are simple and on $\mathrm{Re}(s) = \frac{1}{2}$.*

Corollary 1.4. *Let α and β be as in Theorem 1.1. Then all but finitely many nontrivial zeros of $Z_{Q,\alpha,\beta}(s)$ are simple and on $\mathrm{Re}(s) = \frac{1}{2}$, provided that $\frac{\sqrt{\Delta}}{2a} \geqslant 1$.*

Corollary 1.4 implies that for any real polynomial α with $\deg(\alpha) \geqslant 1$, all but finitely many nontrivial zeros of

$$Z_{u^2+v^2,\alpha,1}(s) = 2\alpha(s)\zeta(2s) + 2\alpha(1-s)\sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1) + \frac{4\pi^s}{\Gamma(s)}\sum_{n=1}^{\infty} n^{s-1/2}\sum_{d|n} d^{1-2s}\int_{-\infty}^{\infty} \mathrm{e}^{-2\pi n\cosh t}\,\mathrm{e}^{(s-1/2)t}\,\mathrm{d}t$$

are simple and on $\mathrm{Re}(s) = \frac{1}{2}$. On the other hand, a really important case is $\alpha(s) = 1$ for $Z_{u^2+v^2,\alpha,1}(s)$. It should be noted that

$$Z_{u^2+v^2,1,1}(s) = Z_{u^2+v^2}(s) = 4\zeta(s)L(s, \chi_{-4}),$$

where $\zeta(s)$ is the Riemann zeta function and χ_{-4} is the Kronecker symbol $(-4/\cdot)$. Thus we may find Corollary 1.4 of some interest concerning the Riemann hypothesis, although Corollary 1.4 cannot say anything about the function $Z_{u^2+v^2,1,1}(s)$ because $\alpha(s) = 1$ and $\beta(s) = 1$ do not satisfy assumption (ii) above.

References

[1] S. Chowla, A. Selberg, On Epstein's zeta-function, J. Reine Angew. Math. 227 (1967) 86–110.

[2] H. Ki, All but finitely many nontrivial zeros of the approximations of the Epstein zeta function are simple and on the critical line, Proc. London Math. Soc. 90 (2005) 321–344.