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C. R. Acad. Sci. Paris, Ser. I 342 (2006) 99–102

COMPTES RENDUS MATHEMATIQUE

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Ordinary Differential Equations

Generalised power series solutions of sub-analytic differential equations

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Received 26 September 2005; accepted after revision 8 November 2005

Available online 6 December 2005

Presented by Étienne Ghys

Abstract

We show that if a solution $y(x)$ of a sub-analytic differential equation admits an asymptotic expansion $\sum_{i=1}^{\infty} c_i x^{\mu_i}$, $\mu_i \in \mathbb{R}^+$, then the exponents μ_i belong to a finitely generated semi-group of $\mathbb{R}^+$. We deduce a similar result for the components of non-oscillating trajectories of real analytic vector fields in dimension n. ***To cite this article: M. Matusinski, J.-P. Rolin, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Séries généralisées solutions d'équations différentielles sous-analytiques. Nous montrons que si une solution $y(x)$ d'une équation différentielle sous-analytique admet un développement asymptotique de la forme $\sum_{i=1}^{\infty} c_i x^{\mu_i}$, $\mu_i \in \mathbb{R}^+$, alors les exposants μ_i appartiennent à un semi-groupe finiment engendré de $\mathbb{R}^+$. Nous en déduisons un résultat analogue pour les composantes des trajectoires non oscillantes de champs de vecteurs analytiques réels en dimension n. ***Pour citer cet article : M. Matusinski, J.-P. Rolin, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction

Let X be an analytic vector field on a real 3-dimensional manifold M. Consider an integral curve $\gamma : t \mapsto \gamma(t)$, $t \geqslant 0$, of X, supposed to be *sub-analytically non-oscillating and transcendental*. That is, any sub-analytic subset of positive codimension of M has a finite number of intersection points with the support $|\gamma|$ of γ. Thus γ has a unique ω-limit point p. The following desingularization theorem is proved in [1]:

Under the previous hypothesis, there exists a so-called γ-admissible transformation $\pi : (\widetilde{M}, \tilde{\gamma}, \tilde{p}) \to (M, \gamma, p)$ *such that the lifted curve is an integral curve of a vector field with non-nilpotent linear part* (*elementary singularity of vector field*).

A γ*-admissible transformation* is a finite sequence of blowing-up transformations with non singular center, and ramified covers. Suppose given a local analytic coordinate system (x, y, z) of M with center p. The non-oscillating

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2005.11.005

assumption allows to suppose that the support $|\gamma|$ belongs to the positive quadrant and that γ is parametrized by x. A key step in the proof is the following result (Proposition 2 of [1]):

Suppose that the axis $x = y = 0$ is not invariant by the vector field X. Consider the projection $(x, y(x))$ and assume that $y(x)$ has an asymptotic expansion $\sum_{i=0}^{\infty} c_i x^{\mu_i}$ ($\mu_i \in \mathbb{R}_+$) with respect to x. Then the exponents μ_i belong to a finitely generated semi-group of $\mathbb{R}_+$.

Note that if the exponents μ_i are rational numbers, the proposition gives a (possibly divergent) asymptotic *Puiseux expansion* of $y(x)$. Consider for example an irrational number $\alpha > 0$ and a solution $H(x)$ of the Euler equation $x^2 y' = y - x$, defined for $x \geqslant 0$. Then $z(x) = xH(x^\alpha)$ is the third component of a trajectory of the vector field defined by $\dot{x} = xy$, $\dot{y} = \alpha y^2$, $\dot{z} = \alpha z - \alpha xy + yz$. The asymptotic expansion of $z(x)$ at the origin is a divergent power series whose exponents are irrational numbers belonging to the semi-group generated by 1 and α.

The main goal of the present Note is to prove a n-dimensional version of the previous result:

Theorem 1.1. *Let X be a analytic vector field on a real analytic n-dimensional manifold M, and γ be sub-analytically non-oscillating and transcendental integral curve of X. Let p be the limit point of γ, and consider a local analytic coordinate system $(x_1, \ldots, x_n)$ with center p, such that $|\gamma|$ is included in the positive octant and γ admits a parametrization $x_1 \mapsto (x_1, x_2(x_1), \ldots, x_n(x_1))$. If any component of γ admits an asymptotic expansion $\sum_{i=1}^{\infty} c_i x_1^{\mu_i}$, then the exponents μ_i belong to a finitely generated semi-group of $\mathbb{R}_+$.*

In [1], this result follows from a two steps elimination process. Assume, *w.l.o.*, $\mu_1 > 2$. The first step shows that the components of $\gamma(x) = (x, y(x), z(x))$ and their derivatives up to order 2 satisfy a system of two analytic equations. The second step uses the hypothesis on the axis $x = y = 0$, in conjunction with a property of analytic mappings (see [4]), to eliminate $z(x)$ between the two equations. Therefore the component $y(x)$ satisfies an analytic differential equation $R(x, y(x), xy'(x), x^2 y''(x)) = 0$. It implies that the exponents μ_i belong to a finitely generated semi-group of $\mathbb{R}_+$. Such a result, which generalizes both [2] and [3], is proved in [1].

Our approach is a generalization in any dimension of this process of an elimination followed by a resolution. The whole elimination step, which is performed in Section 2, does not lead anymore to an analytic differential equation, but to a *sub-analytic differential equation.* The properties of the exponents of a *generalized power series* which is the asymptotic expansion of a solution of a sub-analytic differential equation are investigated in Section 3.

2. From vector fields to sub-analytic differential equations

Proof of Theorem 1.1. The proof follows [1]. Up to a ramification $x \mapsto x^q$, for $q \in \mathbb{N}$ big enough (which would not affect the conclusion of the theorem), we may assume $\mu_1 > n - 1$. With the notation of the introduction, the vector field X is given, in the coordinate system $(x_1, \ldots, x_n)$, by n analytic differential equations, where $\dot{x}_i$ means differentiation with respect to the time t: $\dot{x}_i = a_i(x_1, \ldots, x_n)$, $i = 1, \ldots, n$. These equations obviously imply that the components $x_2(x_1), \ldots, x_n(x_1)$ satisfy the equations $a_1(x_1, \ldots, x_n)x_j' = a_j(x_1, \ldots, x_n)$, $j = 2, \ldots, n$, where x_j' means the derivative with respect to x_1. Let us perform $(n-2)$ times the following operations: compute the derivative of the first equation, and eliminate $x_3'(x_1), \ldots, x_n'(x_1)$ in this equation with the help of remaining ones. We get a system of analytic differential equations $f_j(x_1, x_2(x_1), \ldots, x_2^{(n-1)}(x_1), x_3(x_1), \ldots, x_n(x_1))$, $j = 2, \ldots, n$.

The projection of the analytic subset A of $\mathbb{R}^{2n-1}$ defined in a neighborhood of the origin by the equations $f_2 = \cdots = f_n = 0$ on the space $\mathbb{R}^{n+1} \times \{0\}^{n-2}$ is a sub-analytic set $\pi(A)$. Therefore there exists a sub-analytic function H such that the non-oscillating curve $x_1 \mapsto (x_1, x_2(x_1), \ldots, x_2^{(n-1)}(x_1))$ satisfies the equation $H = 0$. It implies that the function $\varphi : x_1 \mapsto x_1^{-(n-1)} x_2(x_1)$, whose asymptotic expansion is $\sum c_i x_1^{\mu_i - (n-1)}$, is solution of a sub-analytic differential equation $f(x_1, \varphi(x_1), x\varphi'(x_1), \ldots, x^{n-1}\varphi^{(n-1)}(x_1)) = 0$.

Theorem 1.1 is therefore a consequence of the result of the next section. □

3. Generalized power series solutions of sub-analytic differential equations

Theorem 3.1. *Let f be a sub-analytic function defined in a neighborhood of the origin of $\mathbb{R}^{n+2}$. Consider an element φ of a Hardy field at the origin of $\mathbb{R}_+$, solution of the equation $f(x, \varphi(x), x\varphi'(x), \ldots, x^n \varphi^{(n)}(x)) = 0$. If φ admits an asymptotic expansion $\hat{\varphi}(x) = \sum_{i=1}^{\infty} c_i x^{\mu_i}$, $\mu_i \in \mathbb{R}_+^*$, $\lim_{i\to\infty} = +\infty$, then the exponents μ_i belong to a finitely generated additive semi-group of $\mathbb{R}_+^*$.*

Our approach is in some sense an extension of the classical Newton's polygon (more exactly Fine's polygon) method, used in [1–3]. Indeed, we show that the usual transformations of the formal power series, induced by the slopes of such a polygon, reduce the initial sub-analytic equation to an analytic differential equation. Let us recall what happens in the analytic case. For any convergent power series:

$$F(x, u_0, u_1, \dots, u_n) = \sum_{i, j_0, \dots, j_n} F_{i, j_0, \dots, j_n} x^{\alpha_i} u_0^{j_0} \cdots u_n^{j_n},$$

where the exponents α_i belong to a finitely generated semi-group of $\mathbb{R}_+$, and the j_k are positive integers, the classical analysis of the *Newton–Fine's polygon* of F leads to the following conclusion:

- either $F(x, \varphi(x), \dots, x^n\varphi^{(n)}(x)) = 0$ and the exponents μ_k of its expansion $\hat{\varphi}$ belong to a finitely generated semi-group of $\mathbb{R}_+$;
- or else there exists an integer k_0, a positive real number γ and an analytic unit U defined in a neighborhood of the origin such that, if we define φ_1 by $\varphi(x) = \sum_{i=1}^{k_0} c_i x^{\mu_i} + x^{\mu_k}\varphi_1(x)$, then $F(x, \varphi(x), \dots, x^n\varphi^{(n)}(x)) = x^\gamma U(x^{\beta_1}, \dots, x^{\beta_s}, \varphi_1(x), \dots, x^n\varphi_1^{(n)}(x))$, with $\gamma, \beta_1, \dots, \beta_s \in \mathbb{R}_+^*$. In that case, we say that the pair (F, φ) is *monomializable*.

Proof of Theorem 3.1. It relies on the previous Fine's polygon method and on a description of sub-analytic functions, which arises, for example, from [5] or [6]. Consider the sub-analytic function f of the statement of the theorem. It can be described as a finite composition of the three following types of applications:

(i) an analytic function $F : V \to \mathbb{R}$, where V is a neighborhood of the origin of $\mathbb{R}^p$, $p \in \mathbb{N}$,
(ii) a ramification $x \mapsto x^r$, for $x \in \mathbb{R}_+$ and $r \in \mathbb{Q}_+$,
(iii) the division function D defined on $\mathbb{R}^2$ by $D(x, y) = \frac{x}{y}$ if $|y| \geqslant |x|$, and $D(x, y) = 0$ otherwise.

This allows to proceed by induction on the *complexity* of f, defining a sub-analytic function to be *simpler* than f if it is involved in the above description of f. We actually prove that the above dichotomy still holds for pairs (f, φ), where f is a sub-analytic function, which obviously implies the theorem.

1. If f is an analytic function, we already recalled that the dichotomy holds.

Suppose now that f is a sub-analytic function, and that the result has been proved for sub-analytic functions simpler than f.

2. Suppose $f = f_1^r$, with $r \in \mathbb{Q}_+$, and f_1 *simpler* than f. If $f_1(x, \varphi(x), \dots, x^n\varphi^{(n)}(x)) = 0$, we conclude by the induction hypothesis. Otherwise, the pair (f_1, φ) is monomializable, as well as the pair (f, φ).

3. Suppose that $f = F(f_1, \dots, f_l)$, with F analytic and $f_1, \dots, f_l$ are sub-analytic functions simpler than f. If φ is a solution of one of the differential equations $f_j(x, \varphi(x), \dots, x^n\varphi^{(n)}(x)) = 0$, we conclude by the induction hypothesis. Otherwise, it is clear that the pairs $(f_1, \varphi), \dots, (f_l, \varphi)$ are *simultaneously monomializable*. Therefore, φ_1 defined by $\varphi(x) = \sum_{i=1}^{k} c_i x^{\mu_i} + x^{\mu_k}\varphi_1(x)$ is a solution of:

$$F\big(x^{\gamma_1} U_1\big(x, \dots, x^n\varphi_1^{(n)}(x)\big), \dots, x^{\gamma_l} U_l\big(x, \dots, x^n\varphi_1^{(n)}(x)\big)\big) = 0,$$

where $U_1, \dots, U_l$ are analytic and the $\gamma_j{}'s$ belong to $\mathbb{R}_+$. This equation can be written as:

$$F_1\big(x^{\beta_1}, \dots, x^{\beta_s}, \varphi_1(x), x\varphi_1'(x), \dots, x^n\varphi_1^{(n)}(x)\big) = 0,$$

where F_1 is analytic and the $\beta_j{}'s$ belong to $\mathbb{R}_+$.

4. Suppose finally that $f = D(f_1, f_2)$, where f_1, f_2 are simpler than f. Once again, if the pairs (f_1, φ) and (f_2, φ) are simultaneously monomializable, we get:

$$f\big(x, \varphi(x), \dots, x^n\varphi^{(n)}(x)\big) = \frac{x^{\gamma_1} U_1(x, \varphi_1(x), \dots, x^n\varphi_1^{(n)}(x))}{x^{\gamma_2} U_2(x, \varphi_1(x), \dots, x^n\varphi_1^{(n)}(x))}$$

with $\gamma_1 > \gamma_2$ and U_1, U_2 analytic. This shows that the pair (f, φ) is also monomializable. □

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