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Partial Differential Equations/Optimal Control

On the small-time local controllability of a quantum particle in a moving one-dimensional infinite square potential well

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Abstract

We consider a quantum charged particle in a one-dimensional infinite square potential well moving along a line. We control the acceleration of the potential well. The local controllability in large time of this nonlinear control system along the ground state trajectory has been proved recently. We prove that this local controllability does not hold in small time, even if the Schrödinger equation has an infinite speed of propagation. ***To cite this article: J.-M. Coron, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Sur la contrôlabilité en temps petit d'une particule quantique dans un puits de potentiel carré infini unidimensionnel mobile. On considère une particule quantique chargée dans un puits de potentiel carré infini unidimensionnel se déplaçant le long d'une droite. On contrôle l'accélération du puits de potentiel. La contrôlabilité locale autour de l'état fondamental pour des temps grands de ce système de contrôle a été récemment démontrée. Nous montrons que l'on n'a pas contrôlabilité locale pour des temps petits, bien que l'équation de Schrödinger ait une vitesse de propagation infinie. ***Pour citer cet article : J.-M. Coron, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Soit $I=(-1,1)$. On considère le système de contrôle, modélisé par l'équation de Schrödinger,

$$\mathrm{i}\psi_t=-\psi_{xx}-u(t)x\psi,\quad (t,x)\in(0,T)\times I,\qquad \psi(t,-1)=\psi(t,1)=0,\quad t\in(0,T), \tag{1}$$

$$\dot S(t)=u(t),\quad \dot D(t)=S(t),\quad t\in(0,T). \tag{2}$$

C'est un système de contrôle où, au temps $t\in[0,T]$, l'état est $(\psi(t,\cdot),S(t),D(t))\in H_0^1(I;\mathbb{C})\times\mathbb{R}\times\mathbb{R}$ avec $\int_I|\psi(t,x)|^2\,\mathrm{d}x=1$ et le contrôle est $u(t)\in\mathbb{R}$. Ce système a été introduit par Rouchon dans [14]. Il modélise une particule quantique non relativiste chargée dans un puits de potentiel carré infini unidimensionnel se déplaçant le long

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d'une droite Δ. Au temps t, $\psi(t,\cdot)$ est la fonction d'onde de la particule dans un repère lié au puits de potentiel, $S(t)$ est la vitesse du puits de potentiel, $D(t)$ est l'abscisse sur la droite Δ du puits de potentiel et le contrôle $u(t)$ est l'accélération du puits de potentiel. On cherche à contrôler simultanément la fonction d'onde ψ, la vitesse S et la position D du puits de potentiel. Notons que $((\psi,S,D),u) \equiv ((\psi_1,0,0),0)$, avec ψ_1 défini par (11) est une trajectoire du système, i.e. une solution de (1), (2). Dans [2], nous avons montré, en particulier, la contrôlabilité locale de ce système autour de la trajectoire $((\psi_1,0,0),0)$ pour des temps grands. Rappelons que, si on ne s'intéresse pas à (S,D), ce résultat est dû à Beauchard [1]. En particulier on peut aller de $(\psi_1(0,\cdot),0,0)$ à $(\psi_1(T,\cdot),0,\overline{D})$ avec un contrôle petit si $\overline{D}$ est petit, pourvu que T soit assez grand (Theorem 1.1 ci-dessous). Comme la vitesse de propagation de l'équation de Schrödinger est infinie il est naturel de se demander si notre système de contrôle n'est pas localement contrôlable en temps petit le long de la trajectoire $((\psi_1,0,0),0)$. L'objet de cette note est de montrer que ce n'est pas le cas. Plus précisément, on montre le théorème suivant

Théorème 0.1. *Il existe $\varepsilon>0$ tel que, pour tout $\overline{D}\neq 0$, il n'existe pas de $u\in L^2((0,\varepsilon);(-\varepsilon,\varepsilon))$ telle que la solution $(\psi,S,D)\in C^0([0,\varepsilon];H_0^1(I;\mathbb{C}))\times C^0([0,\varepsilon];\mathbb{R})\times C^1([0,\varepsilon];\mathbb{R})$ du problème de Cauchy*

$$\mathrm{i}\psi_t=-\psi_{xx}-u(t)x\psi,\quad (t,x)\in(0,\varepsilon)\times I,\qquad \psi(t,-1)=\psi(t,1)=0,\quad t\in(0,\varepsilon), \tag{3}$$

$$\dot S(t)=u(t),\quad \dot D(t)=S(t),\quad t\in(0,\varepsilon), \tag{4}$$

$$\psi(0,x)=\psi_1(0,x),\quad x\in I,\qquad S(0)=0,\qquad D(0)=0, \tag{5}$$

satisfasse

$$\psi(\varepsilon,x)=\psi_1(\varepsilon,x),\quad x\in I,\qquad S(\varepsilon)=0,\qquad D(\varepsilon)=\overline{D}. \tag{6}$$

Remarque 1. Comme on s'intéresse ici à un problème de contrôlabilité locale autour de la trajectoire $((\psi_1,0,0),0)$, il est naturel de voir ce qui se passe si on remplace (1), (2) par son approximation linéaire le long de cette trajectoire. Cette approximation linéaire est le système de contrôle

$$\mathrm{i}\psi_t=-\psi_{xx}-ux\psi_1,\quad (t,x)\in(0,T)\times I,\qquad \psi(t,-1)=\psi(t,1)=0,\quad t\in(0,T), \tag{7}$$

$$\dot S(t)=u(t),\quad \dot D(t)=S(t),\quad t\in(0,T), \tag{8}$$

avec maintenant $\int_I(\psi(t,x)\bar\psi_1(t,x)+\bar\psi(t,x)\psi_1(t,x))\,\mathrm{d}x=2$. Toutefois il a été montré par Rouchon dans [14] qu'avec (7) à la place de (3), le Théorème 0.1 est faux, ceci bien que le système de contrôle (7), (8) ne soit pas contrôlable (quelque soit le temps de contrôle ; voir de nouveau [14]). Comme nous l'avons rappelé, ci-dessus il est par contre montré dans [2] que le système non linéaire (1), (2) est localement contrôlable le long de la trajectoire $((\psi_1,0,0),0)$ sur $[0,T]$ si T est assez grand. Autrement dit, la non linéarité nous aide pour obtenir de la contrôlabilité mais nous empêche de faire certains déplacements naturels si le temps de contrôle est trop petit.

Pour démontrer le Théorème 0.1, soit $u\in L^2((0,\varepsilon);(-\varepsilon,\varepsilon))$ telle que la solution

$$(\psi,S,D)\in C^0\big([0,\varepsilon];H_0^1(I;\mathbb{C})\big)\times C^0\big([0,\varepsilon];\mathbb{R}\big)\times C^1\big([0,\varepsilon];\mathbb{R}\big)$$

du problème de Cauchy (3)–(5) satisfasse (13). On introduit la fonction

$$V(t):=-\mathrm{i}+\mathrm{i}\int_I \mathrm{e}^{\mathrm{i}\lambda_1 t}\psi(t,x)\varphi_1(x)\,\mathrm{d}x,\quad t\in[0,\varepsilon], \tag{9}$$

où ψ est solution de (3) et où λ_1 est défini dans (11). Après quelques calculs, on montre qu'il existe une constante $C_0>0$ tel que, pour tout $\varepsilon\in[0,1]$,

$$\big|V(\varepsilon)-\|S\|^2_{L^2(0,\varepsilon)}\big|\leqslant C_0\varepsilon\|S\|^2_{L^2(0,\varepsilon)}. \tag{10}$$

On prend $\varepsilon\in(0,1/C_0)\cap(0,1]$. On a alors

$$(u\not\equiv 0)\Rightarrow\big(V(\varepsilon)\neq 0\big)\Rightarrow\big(\psi(\varepsilon,\cdot)\neq\psi_1(\varepsilon,\cdot)\big),$$

ce qui nous donne le Théorème 0.1.

1. Introduction

Let $I = (-1, 1)$. We consider the Schrödinger control system (1), (2). This is a control system, where, at time $t \in [0, T]$, the state is $(\psi(t, \cdot), S(t), D(t)) \in H_0^1(I; \mathbb{C}) \times \mathbb{R} \times \mathbb{R}$ with $\int_I |\psi(t, x)|^2 \, dx = 1$ and the control is $u(t) \in \mathbb{R}$. This system has been introduced by Rouchon in [14]. It models a nonrelativistic quantum charged particle in a one-dimensional infinite square potential well moving along a line. At time t, $\psi(t, \cdot)$ is the wave function of the particle in a frame attached to the potential well, $S(t)$ is the speed of the potential well and $D(t)$ is the displacement of the potential well. The control $u(t)$ is the acceleration of the potential well at time t. We want to control at the same time the wave function ψ of the particle, the speed S and the position D of the potential well. Let

$$\varphi_1(x) := \cos(\pi x/2), \quad \lambda_1 = \pi^2/4, \quad \psi_1(t, x) := \varphi_1(x)\, e^{-i\lambda_1 t}. \tag{11}$$

Then $((\psi, S, D), u) \equiv ((\psi_1, 0, 0), 0)$ is a trajectory of our control system, i.e. a solution of (1), (2). In [2] we have proved, in particular, the local controllability around this trajectory *for large time* for suitable Sobolev spaces (if one does not want to control (S, D), this result is due to Beauchard [1]). The proof of this controllability result relies on the return method (as in [1,3–6,9–13]), quasi-static deformations as in [1,6], power series expansions as in [8], and a Nash–Moser theorem as in [1]. It is probably only the quasi-static deformations parts which require a large time to prove the local controllability (for example, Theorem 8 in [2] probably holds for every $T^* > 0$). As a corollary of [2], one gets the following theorem.

Theorem 1.1. *For every $\varepsilon > 0$, there exist $T > 0$ and $\eta > 0$ such that, for every $\overline{D} \in (-\eta, \eta)$, there exists $u \in L^2((0, T); (-\varepsilon, \varepsilon))$, and (ψ, S, D, u) satisfying* (1), (2) *such that*

$$\big(\psi(0), S(0), D(0)\big) = \big(\psi_1(0, \cdot), 0, 0\big) \quad \text{and} \quad \big(\psi(T), S(T), D(T)\big) = \big(\psi_1(T, \cdot), 0, \overline{D}\big).$$

Since the speed of propagation of the Schrödinger equation is infinite, one may wonder if one can take T arbitrarily small in Theorem 1.1, i.e. if one can replace in that theorem "For every $\varepsilon > 0$, there exist $T > 0$ and $\eta > 0$ such that, ..." by "For every $\varepsilon > 0$ and for every $T > 0$, there exists $\eta > 0$ such that, ...". We prove here that this is not the case. More precisely, we prove the following theorem.

Theorem 1.2. *There exists $\varepsilon > 0$, such that, for every $\overline{D} \neq 0$, there is no $u \in L^2((0, \varepsilon); \mathbb{R})$ satisfying*

$$\big|u(t)\big| \leqslant \varepsilon, \quad t \in (0, \varepsilon) \tag{12}$$

and such that the solution $(\psi, S, D) \in C^0([0, \varepsilon]; H_0^1(I; \mathbb{C})) \times C^0([0, \varepsilon]; \mathbb{R}) \times C^1([0, \varepsilon]; \mathbb{R})$ of the Cauchy problem (3)–(5) *satisfies* (6).

Remark 1. Since we are looking for local statements, it seems natural to look first at the case where one replaces (1), (2) by its linear approximation around the trajectory $((\psi_1, 0, 0), 0)$. This linear approximation is the control system (7), (8), with now $\int_I (\psi(t, x)\bar{\psi}_1(t, x) + \bar{\psi}(t, x)\psi_1(t, x)) \, dx = 2$. However, it has been proved by Rouchon in [14] that with (7) instead of (1) Theorem 1.1 holds with "For every $\varepsilon > 0$, there exist $T > 0$ and $\eta > 0$ such that, ..." replaced by "For every $\varepsilon > 0$ and for every $T > 0$, there exists $\eta > 0$ such that, ...". Note that the linear control system (7), (8) is not controllable around $((\psi_1, 0, 0), 0)$ on $[0, T]$, whatever is $T > 0$, as proved also in [14]. But, as we have mentioned above, it is proved in [2], that the nonlinear control system (1), (2) is locally controllable around the trajectory $((\psi_1, 0, 0), 0)$ on $[0, T]$ if T is large enough. So, in some sense, the nonlinearity helps us to get local controllability but prevents from doing some specific natural motions if the time T is too small, motions which are possible for the linear control system (2), (7) even if $T > 0$ is small.

2. Proof of Theorem 1.2

Let $\varepsilon \in (0, 1]$. Let $u \in L^2((0, \varepsilon); \mathbb{R})$ be such that (12) holds. Let $(\psi, S, D) \in C^0([0, \varepsilon]; H_0^1(I; \mathbb{C})) \times C^0([0, \varepsilon]; \mathbb{R}) \times C^1([0, \varepsilon]; \mathbb{R})$ be the solution of the Cauchy problem (3)–(5). We assume that

$$S(\varepsilon) = 0. \tag{13}$$

Let $\theta \in C^0([0,\varepsilon]; H_0^1(I;\mathbb{C}))$ be defined by

$$\theta(t,x) := e^{i\lambda_1 t}\psi(t,x), \quad (t,x) \in (0,\varepsilon) \times I. \tag{14}$$

From (3), (5), (11) and (14), we have

$$\theta_t = i\theta_{xx} + i\lambda_1\theta + iux\theta, \quad (t,x) \in (0,\varepsilon) \times I, \qquad \theta(t,-1) = \theta(t,1) = 0, \quad t \in (0,\varepsilon), \tag{15}$$

$$\theta(0,x) = \varphi_1(x), \quad x \in I. \tag{16}$$

Let $V \in C^0([0,\varepsilon];\mathbb{C})$ be defined by

$$V(t) := -i + i\int_I \theta(t,x)\varphi_1(x)\,dx, \quad t \in [0,\varepsilon]. \tag{17}$$

From (11), (16) and (17), we have

$$V(0) = 0. \tag{18}$$

From (11), (15) and (17), we get, with integrations by parts,

$$\dot V = -u\int_I \theta x\varphi_1\,dx. \tag{19}$$

Let $V_1 \in C^0([0,\varepsilon];\mathbb{C})$ be defined by

$$V_1(t) := -\int_I \theta(t,x)x\varphi_1(x)\,dx, \quad t \in [0,\varepsilon]. \tag{20}$$

From (11), (15) and (20), we get, with integrations by parts,

$$\dot V_1 = -2i\int_I \theta\varphi_{1x}\,dx - iu\int_I \theta x^2\varphi_1\,dx. \tag{21}$$

From (4), (5), (13), (18)–(21) and integrations by parts, one gets

$$V(\varepsilon) = \int_0^\varepsilon S(t)V_{20}(t)\,dt + \int_0^\varepsilon S(t)^2 V_{21}(t)\,dt, \tag{22}$$

where $V_{20} \in C^0([0,\varepsilon];\mathbb{C})$ and $V_{21} \in L^\infty((0,\varepsilon);\mathbb{C})$ are defined by

$$V_{20}(t) := 2i\int_I \theta(t,x)\varphi_{1x}(x)\,dx, \quad V_{21}(t) := -\frac{i}{2}\int_I \theta_t(t,x)x^2\varphi_1(x)\,dx, \quad t \in [0,\varepsilon]. \tag{23}$$

Let us first estimate $V_{20}(t)$. Let $f \in C^0([0,\varepsilon]; H_0^1(I;\mathbb{C}))$ be defined by

$$f(t,x) := \varphi_1(x)\,e^{ixS(t)}, \quad (t,x) \in [0,\varepsilon] \times I. \tag{24}$$

Let $r \in C^0([0,\varepsilon]; H_0^1(I;\mathbb{C}))$ be defined by

$$r(t,x) := \theta(t,x) - f(t,x), \quad (t,x) \in [0,\varepsilon] \times I. \tag{25}$$

From (4), (5), (15), (16), (24) and (25), we get

$$r_t = ir_{xx} + i\lambda_1 r + iuxr - 2S\varphi_{1x}\,e^{ixS} - i\varphi_1 S^2\,e^{ixS}, \quad (t,x) \in (0,\varepsilon) \times I, \tag{26}$$

$$r(t,-1) = r(t,1) = 0, \quad t \in (0,\varepsilon), \tag{27}$$

$$r(0,x) = 0, \quad x \in I. \tag{28}$$

From (12), (26), (27) and (28), we get

$$\left\|r(t,\cdot)\right\|_{L^2(I)} \leqslant 2\|\varphi_{1x}\|_{L^2(I)} \int_0^t \left|S(\tau)\right| \mathrm{d}\tau + \|\varphi_1\|_{L^2(I)} \int_0^t S^2(\tau)\,\mathrm{d}\tau \leqslant C\varepsilon^{1/2}\|S\|_{L^2(0,\varepsilon)}, \quad t \in [0,\varepsilon]. \tag{29}$$

In (29) and in the following, C denotes various positive constants which may vary from line to line but are independent of $t \in [0,\varepsilon]$, $\varepsilon \in (0,1]$, and $u \in L^2(0,\varepsilon)$ satisfying (12).

From (12) and (24), we readily get

$$\left|\int_I f(t,x)\varphi_{1x}(x)\,\mathrm{d}x + \frac{\mathrm{i}S(t)}{2}\right| \leqslant C\varepsilon\left|S(t)\right|, \quad t \in [0,\varepsilon]. \tag{30}$$

Therefore, using (23), (29) and (30), we have

$$\left|V_{20}(t) - S(t)\right| \leqslant C\left(\varepsilon^{1/2}\|S\|_{L^2(0,\varepsilon)} + \varepsilon\left|S(t)\right|\right), \quad t \in [0,\varepsilon]. \tag{31}$$

Let us now estimate $V_{21}(t)$. From (11), (15), (23) and integrations by parts, we have

$$2V_{21} = \int_I \theta(2\varphi_1 + 4x\varphi_{1x}) + u\theta x^3\varphi_1\,\mathrm{d}x. \tag{32}$$

From (12), (24), (25) and (29), we get

$$\left\|\theta(t,\cdot) - \varphi_1\right\|_{L^2(I)} \leqslant C\varepsilon, \quad t \in [0,\varepsilon]. \tag{33}$$

Using (11), one easily checks that

$$\int_I \varphi_1(2\varphi_1 + 4x\varphi_{1x})\,\mathrm{d}x = 0, \tag{34}$$

which, with (12), (32) and (33), leads to

$$\left|V_{21}(t)\right| \leqslant C\varepsilon, \quad t \in [0,\varepsilon]. \tag{35}$$

From (22), (31) and (35), one gets the existence of $C_0 > 0$ independent of $\varepsilon \in (0,1]$ and of $u \in L^2(0,\varepsilon)$ satisfying (12) such that

$$\left|V(\varepsilon) - \|S\|^2_{L^2(0,\varepsilon)}\right| \leqslant C_0\varepsilon\|S\|^2_{L^2(0,\varepsilon)}. \tag{36}$$

Let us assume that $\varepsilon \in (0, 1/C_0) \cap (0,1]$. Then,

$$(u \not\equiv 0) \Rightarrow \big(V(\varepsilon) \neq 0\big) \Rightarrow \big(\theta(\varepsilon,\cdot) \neq \varphi_1\big) \Rightarrow \big(\psi(\varepsilon,\cdot) \neq \psi_1(\varepsilon,\cdot)\big).$$

This ends the proof of Theorem 1.2.

Let us end this Note by pointing out that more careful estimates on r lead to the following theorem.

Theorem 2.1. *Let $T > 0$ be such that*

$$T < \frac{1}{\pi}\left(\frac{1}{3} - \frac{2}{\pi^3}\right)^{-1/2}. \tag{37}$$

Then, there exists $\varepsilon > 0$ such that, for every $\overline{D} \neq 0$, there is no $u \in L^2((0,T);(-\varepsilon,\varepsilon))$ such that the solution $(\psi,S,D) \in C^0([0,\varepsilon];H^1_0(I;\mathbb{C})) \times C^0([0,\varepsilon];\mathbb{R}) \times C^1([0,\varepsilon];\mathbb{R})$ of the Cauchy problem

$$\mathrm{i}\psi_t = -\psi_{xx} - u(t)x\psi, \quad (t,x) \in (0,T) \times I, \qquad \psi(t,-1) = \psi(t,1) = 0, \quad t \in (0,T), \tag{38}$$

$$\dot{S}(t) = u(t), \quad \dot{D}(t) = S(t), \quad t \in (0,T), \tag{39}$$

$$\psi(0,x) = \psi_1(0,x), \quad x \in I, \qquad S(0) = 0, \qquad D(0) = 0, \tag{40}$$

satisfies

$$\psi(T,x) = \psi_1(T,x), \quad x \in I, \qquad S(T) = 0, \qquad D(T) = \overline{D}. \tag{41}$$

Remark 2. Using analogies of the control system (1), (2) with the toy model given in [1, pp. 853–854] and in [7, Section 3], we conjecture that Theorem 2.1 holds with (37) replaced by the weaker statement

$$T < \frac{4}{\pi} \tag{42}$$

and that (42) is optimal in the sense that, for every $T > 4/\pi$ and for every $\varepsilon > 0$, there exists $\eta > 0$ such that, for every $\overline{D} \in [-\eta, \eta]$, there exists $u \in L^2((0,T); (-\varepsilon, \varepsilon))$ such that the solution $(\psi, S, D) \in C^0([0,T]; H_0^1(I;\mathbb{C})) \times C^0([0,T];\mathbb{R}) \times C^1([0,T];\mathbb{R})$ of the Cauchy problem (38)–(40) satisfies (41).

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