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Proof of the Kurlberg–Rudnick rate conjecture

Shamgar Gurevich, Ronny Hadani

School of Mathematical Sciences, Tel Aviv University, 69978 Tel Aviv, Israel

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Abstract

In this Note we present a proof of the *Hecke quantum unique ergodicity conjecture* for the Berry–Hannay model, a model of quantum mechanics on a two dimensional torus. This conjecture was stated in Z. Rudnick's lectures at MSRI, Berkeley, 1999 and ECM, Barcelona, 2000. ***To cite this article: S. Gurevich, R. Hadani, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Démonstration de la conjecture du taux de Kurlberg–Rudnick. Nous proposons une démonstration de la conjecture d'unique ergodicité quantique d'Hecke pour le modèle de Berry–Hannay, un modèle de mécanique quantique sur un tore de dimension deux. Cette conjecture a été proposée par Z. Rudnick à MSRI, Berkeley, 1999 et à l'ECM, Barcelona, 2000. ***Pour citer cet article : S. Gurevich, R. Hadani, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction

Hannay–Berry model. In 1980 the physicists Hannay and Berry [4] explore a model for quantum mechanics on the two dimensional symplectic torus $(\mathbf{T}, \omega)$.

Quantum chaos. Consider the ergodic discrete dynamical system on the torus, which is generated by an hyperbolic automorphism $A \in \mathrm{SL}_2(\mathbb{Z})$. Quantizing the system, we replace: the classical phase space $(\mathbf{T}, \omega)$ by a Hilbert space $\mathcal{H}_\hbar$, classical observables, i.e., functions $f \in C^\infty(\mathbf{T})$, by operators $\pi_\hbar(f) \in \mathrm{End}(\mathcal{H}_\hbar)$ and classical symmetries by a unitary representation $\rho_\hbar : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{U}(\mathcal{H}_\hbar)$. A fundamental meta-question in the area of quantum chaos is to *describe* the spectral properties of the quantum system $\rho_\hbar(A)$, at least in the semi-classical limit as $\hbar \to 0$.

The rate conjecture. In [5] Kurlberg and Rudnick proved that eigenvectors that satisfy certain additional symmetries of $\rho_\hbar(A)$ are semi-classically equidistributed with respect to the Haar measure on $\mathbf{T}$. In this paper we prove (see Theorem 4) the Kurlberg–Rudnick conjecture [7,8] on the rate of convergence of the relevant distribution to the Haar measure.

E-mail addresses: shamgar@math.tau.ac.il (S. Gurevich), hadani@post.tau.ac.il (R. Hadani).

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2. Classical torus

Let $(\mathbf{T}, \omega)$ be the two dimensional symplectic torus. Together with its linear symplectomorphisms $\Gamma \backsimeq \mathrm{SL}_2(\mathbb{Z})$ it serves as a simple model of classical mechanics (a compact version of the phase space of the harmonic oscillator). More precisely, let $\mathbf{T} = \mathrm{W}/\Lambda$ where W is a two dimensional real vector space and Λ is a rank two unimodular lattice in W. We denote by $\Lambda^* \subseteq \mathrm{W}^*$ the dual lattice, i.e., $\Lambda^* = \{\xi \in \mathrm{W}^* \mid\ \xi(\Lambda) \subset \mathbb{Z}\}$. The lattice Λ^* is identified with the lattice of characters of $\mathbf{T}$ by the map $\xi \in \Lambda^* \mapsto \mathrm{e}^{2\pi\mathrm{i}\langle\xi,\cdot\rangle} \in \mathbf{T}^\vee$, where $\mathbf{T}^\vee := \mathrm{Hom}(\mathbf{T}, \mathbb{C}^*)$.

Classical mechanical system. We consider a very simple discrete mechanical system. An hyperbolic element $A \in \Gamma$, i.e., $|\operatorname{Tr}(A)| > 2$, generates an ergodic discrete dynamical system on $\mathbf{T}$.

3. Quantization of the torus

The Weyl quantization model. The Weyl quantization model works as follows. Let $\mathcal{A}_\hbar$ be a one parameter deformation of the algebra $\mathcal{A}$ of trigonometric polynomials on the torus. This algebra is known in the literature as the Rieffel torus [6]. The algebra $\mathcal{A}_\hbar$ is constructed by taking the free algebra over $\mathbb{C}$ generated by the symbols $\{s(\xi) \mid \xi \in \Lambda^*\}$ and quotient out by the relation $s(\xi+\eta) = \mathrm{e}^{\pi\mathrm{i}\hbar\omega(\xi,\eta)}s(\xi)s(\eta)$. Here ω is the form on W^* induced by the original form ω on W. The algebra $\mathcal{A}_\hbar$ contains as a standard basis the lattice Λ^*. Therefore, one can identify the algebras $\mathcal{A}_\hbar \simeq \mathcal{A}$ as vector spaces. Hence, every function $f \in \mathcal{A}$ can be viewed as an element of $\mathcal{A}_\hbar$. For a fixed $\hbar$ a representation $\pi_\hbar : \mathcal{A}_\hbar \to \mathrm{End}(\mathcal{H}_\hbar)$ serves as a quantization protocol.

Equivariant Weyl quantization of the torus. The group Γ acts on the lattice Λ^*, therefore it acts on $\mathcal{A}_\hbar$. For an element $B \in \Gamma$, we denote by $f \mapsto f^B$ the action of B on an element $f \in \mathcal{A}_\hbar$. Let $\Gamma_p \backsimeq \mathrm{SL}_2(\mathbb{F}_p)$ denotes the quotient group of Γ modulo p.

Theorem 3.1 (Canonical equivariant quantization). *Let $\hbar = \frac{1}{p}$, where p is an odd prime. There exists a unique* (*up to isomorphism*) *pair of representations $\pi_\hbar : \mathcal{A}_\hbar \to \mathrm{End}(\mathcal{H}_\hbar)$ and $\rho_\hbar : \Gamma \to \mathrm{GL}(\mathcal{H}_\hbar)$ satisfying the compatibility condition* (*Egorov identity*) *$\rho_\hbar(B)\pi_\hbar(f)\rho_\hbar(B)^{-1} = \pi_\hbar(f^B)$, where $\pi_\hbar$ is an irreducible representation and $\rho_\hbar$ is a representation of Γ that factors through the quotient group Γ_p.*

Quantum mechanical system. Let $(\pi_\hbar, \rho_\hbar, \mathcal{H}_\hbar)$ be the canonical equivariant quantization. Let A be our fixed hyperbolic element, considered as an element of Γ_p. The element A generates a quantum dynamical system. For every (pure) quantum state $v \in S(\mathcal{H}_\hbar) = \{v \in \mathcal{H}_\hbar\colon \|v\| = 1\}$, $v \mapsto v^A := \rho_\hbar(A)v$.

4. Hecke quantum unique ergodicity

Denote by T_A the centralizer of A in $\Gamma_p \backsimeq \mathrm{SL}_2(\mathbb{F}_p)$. We call T_A the *Hecke torus* (cf. [5]). The precise statement of the *Kurlberg–Rudnick conjecture* (cf. [1] and [7,8]) is given in the following theorem:

Theorem 4.1 (Hecke quantum unique ergodicity). *Let $\hbar = \frac{1}{p}$, p an odd prime. For every $f \in \mathcal{A}_\hbar$ and $v \in S(\mathcal{H}_\hbar)$, we have*:

$$\left|\mathbf{Av}_{\mathrm{T}_A}\bigl(\langle v|\pi_\hbar(f)v\rangle\bigr) - \int\limits_{\mathbf{T}} f\omega\right| \leqslant \frac{C_f}{\sqrt{p}}, \tag{1}$$

where $\mathbf{Av}_{\mathrm{T}_A}(\langle v|\pi_\hbar(f)v\rangle) := \sum_{B\in\mathrm{T}_A}\langle v|\pi_\hbar(f^B)v\rangle$ is the average with respect to the group T_A and C_f is an explicit constant depending only on f.

5. Proof of the Hecke quantum unique ergodicity conjecture

It is enough to prove the conjecture for the case when f is a non-trivial character $\xi \in \Lambda^*$ and v is an Hecke eigenvector with eigencharacter $\chi : \mathrm{T}_A \to \mathbb{C}^*$. In this case Theorem 4.1 can be restated in the form:

Theorem 5.1 (Hecke quantum unique ergodicity (restated)). *Let $\hbar = \frac{1}{p}$, where p is an odd prime. For every $\xi \in \Lambda^*$ and every character $\chi : \mathrm{T}_A \to \mathbb{C}^*$ the following holds*:

$$\left|\sum_{B\in \mathrm{T}_A} \mathrm{Tr}\big(\rho_\hbar(B)\pi_\hbar(\xi)\big)\chi(B)\right| \leqslant 2\sqrt{p}.$$

The trace function. Denote by F the function $F:\Gamma\times\Lambda^*\to\mathbb{C}$ defined by $F(B,\xi)=\mathrm{Tr}(\rho(B)\pi_\hbar(\xi))$. We denote by $\mathrm{V}:=\Lambda^*/p\Lambda^*$ the quotient vector space, i.e., $\mathrm{V}\simeq\mathbb{F}_p^2$. The symplectic form ω specializes to give a symplectic form on V. The group Γ_p is the group of linear symplectomorphisms of V, i.e., $\Gamma_p=\mathrm{Sp}(\mathrm{V},\omega)$. Set $Y_0:=\Gamma\times\Lambda^*$ and $Y:=\Gamma_p\times\mathrm{V}$. We have a natural quotient map $Y_0\to Y$.

Lemma 5.2. *The function $F:Y_0\to\mathbb{C}$ factors through the quotient Y.*

From now on Y will be considered as the default domain of the function F. The function $F:Y\to\mathbb{C}$ is invariant with respect to the action of Γ_p on Y given by the following formula:

$$\begin{aligned}\Gamma_p\times Y &\xrightarrow{\alpha} Y,\\ \big(S,(B,\xi)\big) &\longrightarrow \big(SBS^{-1},S\xi\big).\end{aligned}\tag{2}$$

Geometrization (sheafification). Next, we will phrase a geometric statement that will imply Theorem 5.1. Moving into the geometric setting, we replace the set Y by an algebraic variety and the functions F and χ by sheaf theoretic objects, also of a geometric flavor.

Step 1. The set Y is the set of rational points of an algebraic variety $\mathbb{Y}$ defined over $\mathbb{F}_p$. To be more precise, $\mathbb{Y}\simeq\mathbb{S}\mathrm{p}\times\mathbb{V}$. The variety $\mathbb{Y}$ is equipped with an endomorphism $\mathrm{Fr}:\mathbb{Y}\to\mathbb{Y}$ called Frobenius. The set Y is identified with the set of fixed points of Frobenius $Y=\mathbb{Y}^{\mathrm{Fr}}=\{y\in\mathbb{Y}:\ \mathrm{Fr}(y)=y\}$. Finally, we denote by α the algebraic action of $\mathbb{S}\mathrm{p}$ on the variety $\mathbb{Y}$ (cf. (2)).

Step 2. The following theorem proposes an appropriate sheaf theoretic object standing in place of the function $F:Y\to\mathbb{C}$. Denote by $\mathcal{D}^b_{\mathrm{c,w}}(\mathbb{Y})$ the bounded derived category of constructible ℓ-adic Weil sheaves on $\mathbb{Y}$.

Theorem 5.3 (Geometrization theorem). *There exists an object $\mathcal{F}\in\mathcal{D}^b_{\mathrm{c,w}}(\mathbb{Y})$ satisfying the following properties*:

(i) (*Function*) *It is associated, via the sheaf-to-function correspondence, to the function $F:Y\to\mathbb{C}$, i.e., $f^{\mathcal{F}}=F$.*
(ii) (*Weight*) *It is of weight $w(\mathcal{F})\leqslant 0$.*
(iii) (*Equivariance*) *For every element $S\in\mathbb{S}\mathrm{p}$ there exists an isomorphism $\alpha_S^*\mathcal{F}\simeq\mathcal{F}$.*
(iv) (*Formula*) *On introducing coordinates $\mathbb{V}\simeq\mathbb{A}^2$ we identify $\mathbb{S}\mathrm{p}\simeq\mathbb{SL}_2$. Then there exists an isomorphism $\mathcal{F}_{|\mathbb{T}\times\mathbb{V}}\simeq\mathscr{L}_{\psi(\frac{1}{2}\lambda\mu\frac{a+1}{a-1})}\otimes\mathscr{L}_{\sigma(a)}$.*[1]
Here $\mathbb{T}:=\left\{\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix}\right\}$ stands for the standard torus, (λ,μ) are the coordinates on $\mathbb{V}$ and $\mathscr{L}_\psi$, $\mathscr{L}_\sigma$ the Artin-Schreier and Kummer sheaves.

Geometric statement. Fix an element $\xi\in\Lambda^*$ with $\xi\neq 0$. We denote by i_ξ the inclusion map $i_\xi:\mathrm{T}_A\times\xi\to Y$. Going back to Theorem 5.1 and putting its content in a functorial notation, we write the following inequality: $|pr_!(i_\xi^*(F)\cdot\chi)|\leqslant 2\sqrt{p}$. In words, taking the function $F:Y\to\mathbb{C}$ and restricting F to $\mathrm{T}_A\times\xi$ and get $i_\xi^*(F)$. Multiply i_ξ^*F by the character χ to get $i_\xi^*(F)\cdot\chi$. Integrate $i_\xi^*(F)\cdot\chi$ to the point, this means to sum up all its values, and get a scalar $a_\chi:=pr_!(i_\xi^*(F)\cdot\chi)$. Here pr stands for the projection $pr:\mathrm{T}_A\times\xi\to pt$. Then Theorem 5.1 asserts that the scalar a_χ is of an absolute value less than $2\sqrt{p}$.

Repeat the same steps in the geometric setting. We denote again by i_ξ the closed imbedding $i_\xi:\mathbb{T}_A\times\xi\to\mathbb{Y}$. Take the sheaf $\mathcal{F}$ on $\mathbb{Y}$ and apply the following sequence of operations. Pull-back $\mathcal{F}$ to the closed subvariety $\mathbb{T}_A\times\xi$ and get the sheaf $i_\xi^*(\mathcal{F})$. Take the tensor product of $i_\xi^*(\mathcal{F})$ with the Kummer sheaf $\mathscr{L}_\chi$ and get $i_\xi^*(\mathcal{F})\otimes\mathscr{L}_\chi$. Integrate $i_\xi^*(\mathcal{F})\otimes\mathscr{L}_\chi$ to the point and get the sheaf $pr_!(i_\xi^*(\mathcal{F})\otimes\mathscr{L}_\chi)$ on the point.

Recall $w(\mathcal{F})\leqslant 0$. Knowing that the Kummer sheaf has weight $w(\mathscr{L}_\chi)\leqslant 0$ we deduce that $w(i_\xi^*(\mathcal{F})\otimes\mathscr{L}_\chi)\leqslant 0$.

[1] By this we mean that $\mathcal{F}_{|\mathbb{T}\times\mathbb{V}}$ is isomorphic to the extension of the sheaf defined by the formula in the right-hand side.

Theorem 5.4 (Deligne, Weil II [2]). *Let $\pi : \mathbb{X}_1 \to \mathbb{X}_2$ be a morphism of algebraic varieties. Let $\mathcal{L} \in \mathcal{D}^b_{\mathrm{c,w}}(\mathbb{X}_1)$ be a sheaf of weight $w(\mathcal{L}) \leqslant w$ then $w(\pi_!(\mathcal{L})) \leqslant w$.*

Using Theorem 5.4 we get $w(pr_!(i_\xi^*(\mathcal{F}) \otimes \mathscr{L}_\chi)) \leqslant 0$.

Now, consider the sheaf $\mathcal{G} := pr_!(i_\xi^*(\mathcal{F}) \otimes \mathscr{L}_\chi)$. It is an object in $\mathcal{D}^b_{\mathrm{c,w}}(pt)$. The sheaf $\mathcal{G}$ is associated by *Grothendieck's Sheaf-To-Function correspondence* to the scalar a_χ:

$$a_\chi = \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{Tr}(\mathrm{Fr}|_{\mathrm{H}^i(\mathcal{G})}). \tag{3}$$

Finally, we can give the geometric statement about $\mathcal{G}$, which will imply Theorem 5.1.

Lemma 5.5 (Vanishing Lemma). *Let $\mathcal{G} = pr_!(i_\xi^*(\mathcal{F}) \otimes \mathscr{L}_\chi)$. All cohomologies $\mathrm{H}^i(\mathcal{G})$ vanish except for $i = 1$. Moreover, $\mathrm{H}^1(\mathcal{G})$ is a two dimensional vector space.*

Theorem 5.1 now follows easily. By Lemma 5.5 only the first cohomology $\mathrm{H}^1(\mathcal{G})$ does not vanish and it is two dimensional. Having that $w(\mathcal{G}) \leqslant 0$ implies that the eigenvalues of Frobenius acting on $\mathrm{H}^1(\mathcal{G})$ are of absolute value $\leqslant \sqrt{p}$. Hence, using formula (3) we get $|a_\chi| \leqslant 2\sqrt{p}$.

Proof of the Vanishing Lemma. *Step* 1. All tori in $\mathbb{S}\mathrm{p}$ are conjugated. On introducing coordinates, i.e., $\mathbb{V} \simeq \mathbb{A}^2$, we make the identification $\mathbb{S}\mathrm{p} \simeq \mathbb{SL}_2$. In these terms there exists an element $S \in \mathbb{SL}_2$ conjugating the *Hecke* torus $\mathbb{T}_A \subset \mathbb{SL}_2$ with the standard torus $\mathbb{T} = \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right)\right\} \subset \mathbb{SL}_2$, namely $S\mathbb{T}_A S^{-1} = \mathbb{T}$.

Step 2. Using the equivariance property of the sheaf $\mathcal{F}$ (see Theorem 5.3, property (iii)) we see that it is *sufficient* to prove the Vanishing Lemma for the sheaf $\mathcal{G}_{st} := pr_!(i_\eta^* \mathcal{F} \otimes \alpha_{S!} \mathscr{L}_\chi)$, where $\eta = S \cdot \xi$ and α_S is the restriction of the action α to the element S.

Step 3. The Vanishing Lemma holds for the sheaf $\mathcal{G}_{st}$. We write $\eta = (\lambda, \mu)$. By Theorem 5.3 Property (iv) we have $i_\eta^* \mathcal{F} \simeq \mathscr{L}_{\psi(\frac{1}{2}\lambda\mu\frac{a+1}{a-1})} \otimes \mathscr{L}_{\sigma(a)}$, where a is the coordinate of the standard torus $\mathbb{T}$ and $\lambda \cdot \mu \neq 0$[2]. The sheaf $\alpha_{S!}\mathscr{L}_\chi$ is a character sheaf on the torus $\mathbb{T}$. A direct computation proves the Vanishing Lemma. □

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[2] This is a direct consequence of the fact that $A \in \mathrm{SL}_2(\mathbb{Z})$ is an hyperbolic element and does not have eigenvectors in Λ^*.