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Numerical Analysis

Resolution of the finite Markov moment problem ☆

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Abstract

We expose in full detail a constructive procedure to invert the so-called 'finite Markov moment problem'. The proofs rely on the general theory of Toeplitz matrices together with the classical Newton's relations. ***To cite this article: L. Gosse, O. Runborg, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Résolution du problème fini des moments de Markov. Nous présentons en détail une procédure constructive pour inverser le « problème fini des moments de Markov ». Les preuves reposent sur la théorie générale des matrices de Toeplitz et les classiques relations de Newton. ***Pour citer cet article : L. Gosse, O. Runborg, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Version française abrégée

Afin d'inverser le systeme fini et mal conditionné (1), Koborov, Sklyar et Fardigola, [8,12] ont proposé un algorithme recursif non-linéaire. Dans [6] nous avons prouvé un lemme le réduisant a l'extraction de valeurs propres généralisées, voir (4). Cette Note vise a expliquer en détail les raisons pour lesquelles cette procédure simplifiée résout le problème des moments de Markov. Apres avoir rappelé quelques éléments de la théorie des matrices de Toeplitz et les relations de Newton (Propositions 2.1 et 2.2), nous reformulons cet algorithme simplifié afin d'établir facilement certains lemmes techniques. Finalement, le Théorème 2.7 démontre le lien entre valeurs propres généralisées (4) et l'inversion de (1).

1. Introduction

We aim at inverting a moment system often associated with the prestigious name of Markov, as appearing in [2,3,5,10,11] in several fields of application; consult [4,9,13] for general background on moment problems. The original

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problem is the following. Given the *moments* m_k for $k = 1, \dots, K$, find a bounded measurable *density* function f and a real value $X > 0$ such that

(1) $\int_0^X f(\xi)\xi^{k-1}\,\mathrm{d}\xi = m_k$, $k = 1, \dots, K$,
(2) $|f(\xi)| = 1$ almost everywhere on $]0, X[$,
(3) f has no more than $K - 1$ discontinuity points inside $]0, X[$.

The solution is a piecewise constant function taking values in $\{-1, 1\}$ a.e. on $]0, X[$ and changing sign in at most $K - 1$ points, which we denote $\{u_k\}$, ordered such that $0 \leqslant u_1 \leqslant \cdots \leqslant u_K = X$. Finding $\{u_k\}$ from $\{m_k\}$ is an ill-conditioned problem when the u_k values come close to each other; its Jacobian is a Vandermonde matrix and iterative numerical resolution routines require extremely good starting guesses. For less than four moments, however, a direct method based on solving polynomial equations was presented in [11]. Here we are concerned with an arbitrary number of moments $K \in \mathbb{N}$.

We consider a slightly modified version of the problem where f takes values in $\{1, 0\}$ instead of $\{-1, 1\}$ and the moments are scaled as $m_k \to k m_k$. This is the precise setting for the applications in geometrical optics that we are interested in [6]. Moreover, to simplify the discussion we confine ourselves to the case when K is even, setting $K =: 2n$. The resulting problem can then be written as an algebraic system of nonlinear equations: Given m_k find u_k such that

$$m_k = \sum_{j=1}^{n} u_{2j}^k - u_{2j-1}^k, \quad k = 1, \dots, K = 2n. \tag{1}$$

An algorithm for solving this problem was presented by Koborov, Sklyar and Fardigola in [8,12]. It requires solving a sequence of high degree polynomial equations, constructed through a rather complicated process with unclear stability properties. In [6] we showed that the algorithm can be put in a more simple form that makes it much more suitable for numerical implementation. The simplified algorithm reads

(1) Construct the matrices A and B:

$$A = \begin{pmatrix} 1 & & & \\ -m_1 & 2 & & \\ \vdots & \ddots & \ddots & \\ -m_{2n-1} & \cdots & -m_1 & 2n \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & & & \\ m_1 & 2 & & \\ \vdots & \ddots & \ddots & \\ m_{2n-1} & \cdots & m_1 & 2n \end{pmatrix}. \tag{2}$$

(2) Let $\boldsymbol{m} = (m_1, m_2, \dots, m_{2n})^{\mathrm{T}}$ and solve the lower triangular Toeplitz linear systems

$$A\boldsymbol{a} = \boldsymbol{m}, \qquad B\boldsymbol{b} = -\boldsymbol{m}, \tag{3}$$

to get $\boldsymbol{a}$ and $\boldsymbol{b}$.

(3) Construct the matrices A_1, A_2 from $\boldsymbol{a} = (a_1, a_2, \dots, a_{2n})^{\mathrm{T}}$ as

$$A_1 = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{pmatrix}, \qquad A_2 = \begin{pmatrix} a_2 & a_3 & \cdots & a_{n+1} \\ a_3 & a_4 & \cdots & a_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+1} & a_{n+2} & \cdots & a_{2n} \end{pmatrix},$$

and the corresponding matrices B_1, B_2 from $\boldsymbol{b}$.

(4) Compute the generalized eigenvalues of the problems

$$A_2\boldsymbol{v} = uA_1\boldsymbol{v}, \qquad B_2\boldsymbol{v} = uB_1\boldsymbol{v}. \tag{4}$$

The generalized eigenvalues of the left problem are precisely the even u_k-values, $\{u_{2k}\}_{k=1}^n$, and the generalized eigenvalues of the right problem are the odd ones, $\{u_{2k-1}\}_{k=1}^n$.

The forthcoming section is devoted to a complete justification of this algorithm. We recall that these inversion routines have been shown to be numerically efficient in the paper [6].

2. Analysis of the algorithm

We begin by stating two classical results of prime importance for the analysis.

Let $\mathbb{L}^n \subset \mathbb{R}^{n\times n}$ be the set of lower triangular $n \times n$ real Toeplitz matrices. We define the diagonal scaling matrix and the mapping $\mathcal{T}:\mathbb{R}^n \to \mathbb{L}^n$ as

$$\Lambda = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & n-1 \end{pmatrix}, \qquad \mathcal{T}(\boldsymbol{x}) := \begin{pmatrix} x_1 & & & \\ x_2 & x_1 & & \\ \vdots & \ddots & \ddots & \\ x_n & \cdots & x_2 & x_1 \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

The mapping $\mathcal{T}$ has the following properties, see e.g. [1]:

Proposition 2.1. *Lower triangular Toeplitz matrices commute and $\mathbb{L}^n$ is closed under matrix multiplication and* (*when the inverse exists*) *inversion,*

$$\mathcal{T}(\boldsymbol{x})\mathcal{T}(\boldsymbol{y}) = \mathcal{T}(\boldsymbol{y})\mathcal{T}(\boldsymbol{x}) \in \mathbb{L}^n, \qquad \mathcal{T}(\boldsymbol{x})^{-1} \in \mathbb{L}^n. \tag{5}$$

Moreover, $\mathcal{T}$ is linear and

$$\mathcal{T}(\boldsymbol{x})\boldsymbol{y} = \mathcal{T}(\boldsymbol{y})\boldsymbol{x}, \qquad \mathcal{T}(\boldsymbol{x})\mathcal{T}(\boldsymbol{y}) = \mathcal{T}\big(\mathcal{T}(\boldsymbol{x})\boldsymbol{y}\big). \tag{6}$$

The Λ matrix has the property

$$\mathcal{T}(\Lambda\boldsymbol{x}) = \Lambda\mathcal{T}(\boldsymbol{x}) - \mathcal{T}(\boldsymbol{x})\Lambda. \tag{7}$$

Another result that we rely heavily upon is the classical Newton relations, see e.g. [7]:

Proposition 2.2 (Newton's relations). *Let P be the n-degree polynomial,*

$$P(x) = c_0 + c_1 x + \cdots + c_n x^n =: c_n(x - x_1)\cdots(x - x_n).$$

Set $S_0 = n$ and define S_k for $k > 0$ as the sum of the roots of P taken to the power k, $S_k = \sum_{j=1}^n x_j^k$. Then, the $n+1$ following relations hold:

$$c_k S_0 + c_{k+1}S_1 + \cdots + c_n S_{n-k} = k c_k, \quad k = 0,\dots,n. \tag{8}$$

2.1. Reformulation of the simplified algorithm

We want to write the equation $A\boldsymbol{a} = \boldsymbol{m}$ using the mapping $\mathcal{T}$: hence we augment the $\boldsymbol{m}$ and $\boldsymbol{a}$-vectors with a zero and one element, respectively, to get $\widetilde{\boldsymbol{m}} = (0,\ \boldsymbol{m})^{\mathrm{T}}$ and $\tilde{\boldsymbol{a}} = (1,\ \boldsymbol{a})^{\mathrm{T}}$, both in $\mathbb{R}^{K+1}$. We observe that the A-matrix in (2) is the lower right $K \times K$ block of $\Lambda - \mathcal{T}(\widetilde{\boldsymbol{m}})$. Therefore,

$$\big(\Lambda - \mathcal{T}(\widetilde{\boldsymbol{m}})\big)\tilde{\boldsymbol{a}} = -\widetilde{\boldsymbol{m}} + \begin{pmatrix} 0 \\ A\boldsymbol{a} \end{pmatrix} = \begin{pmatrix} 0 \\ A\boldsymbol{a} - \boldsymbol{m} \end{pmatrix}.$$

Thus the equation $A\boldsymbol{a} = \boldsymbol{m}$ in (3) is equivalent to

$$\mathcal{T}(\widetilde{\boldsymbol{m}})\tilde{\boldsymbol{a}} = \Lambda\tilde{\boldsymbol{a}}. \tag{9}$$

By the same argument, $B\boldsymbol{b} = -\boldsymbol{m}$ in (3) is equivalent to

$$\mathcal{T}(\widetilde{\boldsymbol{m}})\tilde{\boldsymbol{b}} = -\Lambda\tilde{\boldsymbol{b}}. \tag{10}$$

with $\tilde{\boldsymbol{b}} = (1, \boldsymbol{b})^{\mathrm{T}}$.

We can then directly also show that $\mathcal{T}(\tilde{\boldsymbol{a}})$ and $\mathcal{T}(\tilde{\boldsymbol{b}})$ are in fact each other's inverses.

Lemma 2.3. *When $\tilde{\boldsymbol{a}}$ are $\tilde{\boldsymbol{b}}$ given by* (9), (10) *then $\mathcal{T}(\tilde{\boldsymbol{a}})\mathcal{T}(\tilde{\boldsymbol{b}}) = I$.*

Proof. By Proposition 2.1 and (9), (10),

$$\Lambda\mathcal{T}(\tilde{\boldsymbol{a}})\tilde{\boldsymbol{b}} = \mathcal{T}(\Lambda\tilde{\boldsymbol{a}})\tilde{\boldsymbol{b}} + \mathcal{T}(\tilde{\boldsymbol{a}})\Lambda\tilde{\boldsymbol{b}} = \mathcal{T}\big(\mathcal{T}(\widetilde{\boldsymbol{m}})\tilde{\boldsymbol{a}}\big)\tilde{\boldsymbol{b}} + \mathcal{T}(\tilde{\boldsymbol{a}})\Lambda\tilde{\boldsymbol{b}} = \mathcal{T}(\tilde{\boldsymbol{a}})\big[\mathcal{T}(\widetilde{\boldsymbol{m}})\tilde{\boldsymbol{b}} + \Lambda\tilde{\boldsymbol{b}}\big] = 0.$$

Thus $\mathcal{T}(\tilde{\boldsymbol{a}})\tilde{\boldsymbol{b}}$ lies in the nullspace of Λ which is spanned by the vector $\mathbf{1} := (1, 0, \dots, 0)^{\mathrm{T}}$. Moreover, $(\mathcal{T}(\tilde{\boldsymbol{a}})\tilde{\boldsymbol{b}})_1 = 1$ since the first elements of $\tilde{\boldsymbol{a}}$ and $\tilde{\boldsymbol{b}}$ are both one and we must in fact have $\mathcal{T}(\tilde{\boldsymbol{a}})\tilde{\boldsymbol{b}} = \mathbf{1}$. The lemma then follows from (6) and the definition of $\mathcal{T}$. □

2.2. *What lies beneath the algorithm*

To understand the workings of the algorithm we need to introduce some new quantities and determine how they relate to $\tilde{\boldsymbol{a}}$, $\tilde{\boldsymbol{b}}$ and $\widetilde{\boldsymbol{m}}$.

Let us start with some notation: we set $x_j = u_{2j}$ and $y_j = u_{2j-1}$ for $j = 1, \dots, n$. Furthermore, we introduce the sums

$$X_k = \sum_{j=1}^{n} x_j^k, \quad Y_k = \sum_{j=1}^{n} y_j^k, \quad k = 1, 2, \dots, K = 2n, \tag{11}$$

and define $X_0 = Y_0 = K$. In the even case, it then holds that

$$m_k = \sum_{j=1}^{n} x_j^k - \sum_{j=1}^{n} y_j^k = X_k - Y_k, \quad k = 1, \dots, K = 2n.$$

We also define the two polynomials

$$p(x) = (x - x_1) \cdots (x - x_n) =: c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} + c_n x^n, \tag{12}$$

and

$$q(x) = (x - y_1) \cdots (x - y_n) =: d_0 + d_1 x + \cdots + d_{n-1}x^{n-1} + d_n x^n. \tag{13}$$

We note here that by construction $c_n = d_n = 1$.

By applying (8) to $x^n p(x)$ with $k = 0, \dots, K$ we get

$$\begin{pmatrix} X_0 & & & \\ X_1 & X_0 & & \\ \vdots & \ddots & \ddots & \\ X_K & \cdots & X_1 & X_0 \end{pmatrix} \begin{pmatrix} c_n \\ c_{n-1} \\ \vdots \\ c_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} K c_n \\ (K-1)c_{n-1} \\ \vdots \\ (K-n)c_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The analogous system of equations holds also for Y_k and d_k. We introduce now some shorthand notation to write these equations in a concise form. First we set $\bar{\boldsymbol{c}} = (c_n, \dots, c_0)^{\mathrm{T}} \in \mathbb{R}^{n+1}$ and $\bar{\boldsymbol{d}} = (d_n, \dots, d_0)^{\mathrm{T}} \in \mathbb{R}^{n+1}$. We then construct the larger vectors, padded with zeros: $\boldsymbol{c} = (\bar{\boldsymbol{c}},\ \mathbf{0})^{\mathrm{T}}$ and $\boldsymbol{d} = (\bar{\boldsymbol{d}},\ \mathbf{0})^{\mathrm{T}}$, both in $\mathbb{R}^{K+1}$. Finally, we let $\boldsymbol{X} = (X_0, \dots, X_K)^{\mathrm{T}}$ and $\boldsymbol{Y} = (Y_0, \dots, Y_K)^{\mathrm{T}}$. Using $\mathcal{T}$ and Λ we can state the systems of equations above as follows:

$$\mathcal{T}(\boldsymbol{X})\boldsymbol{c} = (KI - \Lambda)\boldsymbol{c}, \qquad \mathcal{T}(\boldsymbol{Y})\boldsymbol{d} = (KI - \Lambda)\boldsymbol{d}. \tag{14}$$

We also clearly have $\widetilde{\boldsymbol{m}} = \boldsymbol{X} - \boldsymbol{Y}$.

Before we can relate $\boldsymbol{c}$ and $\boldsymbol{d}$ with $\tilde{\boldsymbol{a}}$ we need the following lemma:

Lemma 2.4. *Let $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $\boldsymbol{f}(\boldsymbol{x}) := \mathcal{T}(\boldsymbol{x})^{-1}\Lambda\boldsymbol{x}$ for $\boldsymbol{x}$ with a non-zero first element. Then $\boldsymbol{f}(\boldsymbol{x}_1) = \boldsymbol{f}(\boldsymbol{x}_2)$ implies that $\boldsymbol{x}_1 = \alpha\boldsymbol{x}_2$ for some non-zero $\alpha \in \mathbb{R}$.*

Proof. Suppose $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{y}$. Then $\mathcal{T}(\boldsymbol{x})\boldsymbol{y} = \Lambda\boldsymbol{x}$ and by Proposition 2.1, $(\mathcal{T}(\boldsymbol{y}) - \Lambda)\boldsymbol{x} = 0$. Hence $\boldsymbol{f}(\boldsymbol{x}_1) = \boldsymbol{f}(\boldsymbol{x}_2)$ implies that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ both lie in the nullspace of $\mathcal{T}(\boldsymbol{y}) - \Lambda$. Since the top left element of Λ is zero, it follows that the first element of $\boldsymbol{y}$ is zero and therefore the diagonal of $\mathcal{T}(\boldsymbol{y})$ is zero. Consequently, the nullspace of $\mathcal{T}(\boldsymbol{y}) - \Lambda$ has the same dimension as that of Λ, which is one. □

We can now merge together and express the general structure from (12), (13) and (9), (10) in the most concise way.

Lemma 2.5. *Suppose $\boldsymbol{c}$, $\boldsymbol{d}$ are defined by* (12), (13) *and $\tilde{\boldsymbol{a}}$, $\tilde{\boldsymbol{b}}$ by* (9), (10). *Then*

$$\mathcal{T}(\tilde{\boldsymbol{a}})\boldsymbol{c} = \boldsymbol{d}, \qquad \mathcal{T}(\tilde{\boldsymbol{b}})\boldsymbol{d} = \boldsymbol{c}.$$

Proof. We only need to prove the left equality. The right one follows immediately from Lemma 2.3. Let $\boldsymbol{v} = \mathcal{T}(\tilde{\boldsymbol{a}})\boldsymbol{c} = \mathcal{T}(\boldsymbol{c})\tilde{\boldsymbol{a}}$. We want to show that $\boldsymbol{v} = \boldsymbol{d}$. We note first that by (14) $\mathcal{T}(\boldsymbol{X})\boldsymbol{c} = (KI - \Lambda)\boldsymbol{c} \Rightarrow \mathcal{T}(\boldsymbol{c})\boldsymbol{X} = (KI - \Lambda)\boldsymbol{c} \Rightarrow \boldsymbol{X} = K\mathcal{T}(\boldsymbol{c})^{-1}\boldsymbol{c} - \mathcal{T}(\boldsymbol{c})^{-1}\Lambda\boldsymbol{c}$, where $\mathcal{T}(\boldsymbol{c})$ is invertible since $c_n = 1$. Moreover, it is clear that $\mathcal{T}(\boldsymbol{y})\boldsymbol{1} = \boldsymbol{y}$ for all $\boldsymbol{y}$. Hence, $\boldsymbol{X} = K\boldsymbol{1} - \mathcal{T}(\boldsymbol{c})^{-1}\Lambda\boldsymbol{c}$. In the same way we also obtain $\boldsymbol{Y} = K\boldsymbol{1} - \mathcal{T}(\boldsymbol{d})^{-1}\Lambda\boldsymbol{d}$. Then,

$$\mathcal{T}(\widetilde{\boldsymbol{m}})\tilde{\boldsymbol{a}} = \mathcal{T}(\tilde{\boldsymbol{a}})\widetilde{\boldsymbol{m}} = \mathcal{T}(\tilde{\boldsymbol{a}})(\boldsymbol{X} - \boldsymbol{Y}) = -\mathcal{T}(\tilde{\boldsymbol{a}})\mathcal{T}(\boldsymbol{c})^{-1}\Lambda\boldsymbol{c} + \mathcal{T}(\tilde{\boldsymbol{a}})\mathcal{T}(\boldsymbol{d})^{-1}\Lambda\boldsymbol{d}.$$

We now note that by Proposition 2.1, $\mathcal{T}(\tilde{\boldsymbol{a}})\mathcal{T}(\boldsymbol{c})^{-1}\Lambda\boldsymbol{c} = \mathcal{T}(\boldsymbol{c})^{-1}\mathcal{T}(\tilde{\boldsymbol{a}})\Lambda\boldsymbol{c} = \mathcal{T}(\boldsymbol{c})^{-1}\mathcal{T}(\Lambda\boldsymbol{c})\tilde{\boldsymbol{a}} = \mathcal{T}(\boldsymbol{c})^{-1}\Lambda\mathcal{T}(\boldsymbol{c})\tilde{\boldsymbol{a}} - \Lambda\tilde{\boldsymbol{a}} = \mathcal{T}(\boldsymbol{c})^{-1}\Lambda\boldsymbol{v} - \Lambda\tilde{\boldsymbol{a}}$. Since also, $\mathcal{T}(\boldsymbol{c})\mathcal{T}(\tilde{\boldsymbol{a}}) = \mathcal{T}(\boldsymbol{v})$ we get $\mathcal{T}(\widetilde{\boldsymbol{m}})\tilde{\boldsymbol{a}} = -\mathcal{T}(\boldsymbol{c})^{-1}\Lambda\boldsymbol{v} + \Lambda\tilde{\boldsymbol{a}} + \mathcal{T}(\tilde{\boldsymbol{a}})\mathcal{T}(\boldsymbol{d})^{-1}\Lambda\boldsymbol{d} = \mathcal{T}(\tilde{\boldsymbol{a}})[\mathcal{T}(\boldsymbol{d})^{-1}\Lambda\boldsymbol{d} - \mathcal{T}(\boldsymbol{v})^{-1}\Lambda\boldsymbol{v}] + \Lambda\tilde{\boldsymbol{a}}$. Consequently, by (9), $\mathcal{T}(\boldsymbol{d})^{-1}\Lambda\boldsymbol{d} = \mathcal{T}(\boldsymbol{v})^{-1}\Lambda\boldsymbol{v}$ and by Lemma 2.4, $\boldsymbol{v} = \alpha\boldsymbol{d}$; for some $\alpha \in \mathbb{R}$. But for the first element in $\boldsymbol{v}$ we then have $v_1 = c_n = \alpha d_n$ and we get $\alpha = 1$ since $c_n = d_n = 1$. □

Finally, we also establish the following lemma.

Lemma 2.6. *Let V and W be the Vandermonde matrices corresponding to $\{x_j\}_{j=0}^n$ and $\{y_j\}_{j=0}^n$ respectively, with $x_0 = y_0 = 0$. Then*

$$V^{\mathrm{T}}R\mathcal{T}(\bar{\boldsymbol{c}})V = \operatorname{diag}\big(\{\tilde{p}'(x_k)\}_{k=0}^n\big), \qquad W^{\mathrm{T}}R\mathcal{T}(\bar{\boldsymbol{d}})W = \operatorname{diag}\big(\{\tilde{q}'(x_k)\}_{k=0}^n\big),$$

where $\tilde{p}(x) := xp(x)$, $\tilde{q}(x) := xq(x)$ and $R = \{\delta_{n+2-i-j}\} \in \mathbb{R}^{n+1\times n+1}$ is the reversion matrix.

Proof. We have

$$\big(V^{\mathrm{T}}R\mathcal{T}(\bar{\boldsymbol{c}})V\big)_{ij} = \sum_{r=0}^{n}\sum_{\ell=0}^{r} c_r x_{i-1}^{\ell} x_{j-1}^{r-\ell} = \begin{cases} \frac{\tilde{p}(x_{i-1}) - \tilde{p}(x_{j-1})}{x_{i-1} - x_{j-1}}, & i \neq j, \\ \tilde{p}'(x_{i-1}), & i = j, \end{cases}$$

showing the left equality since $\{x_j\}_{j=0}^n$ are the roots of $\tilde{p}(x)$. The right equality follows in the same way. □

2.3. Conclusion

We can now conclude and show that the unknown values u_j in (1) are indeed the generalized eigenvalues of (4).

Theorem 2.7. *Suppose $K = 2n$; let $\boldsymbol{a}$, $\boldsymbol{b}$ be defined by* (3). *If all values $\{x_j\} \cup \{y_j\}$ are distinct, then $\{x_j\}$, $\{y_j\}$ are the generalized eigenvalues of* (4).

Proof. Let $\tilde{\boldsymbol{a}}$, $\tilde{\boldsymbol{b}}$ be defined by (9), (10), which is equivalent to (3). Also define $\boldsymbol{c}$ and $\boldsymbol{d}$ as before by (12), (13). By Lemma 2.5 we have $\mathcal{T}(\tilde{\boldsymbol{a}})\boldsymbol{c} = \boldsymbol{d} \in \mathbb{R}^{2n+1}$, i.e.

$$\left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ a_1 & 1 & & & & & & \\ \vdots & \ddots & \ddots & & & & & \\ a_n & \cdots & a_1 & 1 & & & & \\ \hline a_{n+1} & a_n & \cdots & a_1 & 1 & & & \\ a_{n+2} & a_{n+1} & \cdots & a_2 & a_1 & 1 & & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \\ a_{2n} & a_{2n-1} & \cdots & a_n & a_{n-1} & \cdots & a_1 & 1 \end{array}\right) \begin{pmatrix} \bar{\boldsymbol{c}} \\ - \\ 0 \end{pmatrix} = \begin{pmatrix} \bar{\boldsymbol{d}} \\ - \\ 0 \end{pmatrix}. \tag{15}$$

Clearly, the lower left block of the matrix multiplied by $\bar{\boldsymbol{c}}$ is zero, i.e. $\sum_{i=0}^{n} c_i a_{i+k} = 0$ for $k = 1, \ldots, n$. Now, let v_i be the coefficients of the polynomial $v(x) := p(x)/(x - x_j)$ for some fixed j. Hence, by the special structure of (12),

$$c_0 + c_1 x + \cdots + c_n x^n =: (v_1 + v_2 x + \cdots + v_n x^{n-1})(x - x_j),$$

and, for $i = 0, \ldots, n$,

$$c_i = \begin{cases} -x_j v_{i+1}, & i = 0, \\ v_i - x_j v_{i+1}, & 1 \leqslant i \leqslant n-1, \\ v_i, & i = n. \end{cases}$$

Thus we deduce,

$$0 = -x_j v_1 a_k + \sum_{i=1}^{n-1} (v_i - x_j v_{i+1}) a_{i+k} + v_n a_{n+k} = \sum_{i=1}^{n} v_i a_{i+k} - x_j \sum_{i=1}^{n} v_i a_{i+k-1},$$

which is the componentwise statement of $A_2 \boldsymbol{v} = x_j A_1 \boldsymbol{v}$. It remains to show that the rightmost sum is non-zero for at least some k, so that x_j is indeed a well-defined generalized eigenvalue. Let $\bar{\boldsymbol{a}} = (1, a_1, \ldots, a_n)^{\mathrm{T}}$ and $\bar{\boldsymbol{v}} = (0, v_1, \ldots, v_n)^{\mathrm{T}}$. Then (15) gives $\mathcal{T}(\bar{\boldsymbol{a}})\bar{\boldsymbol{c}} = \bar{\boldsymbol{d}}$ and, using Lemma 2.6 while taking $k = 1$ we have the sum

$$\sum_{i=1}^{n} v_i a_i = \bar{\boldsymbol{a}}^{\mathrm{T}} \bar{\boldsymbol{v}} = \left(\mathcal{T}(\bar{\boldsymbol{c}})^{-1} \bar{\boldsymbol{d}}\right)^{\mathrm{T}} \bar{\boldsymbol{v}} = (V^{\mathrm{T}} R \bar{\boldsymbol{d}})^{\mathrm{T}} \operatorname{diag}\left(\left\{\tilde{p}'(x_k)^{-1}\right\}\right) V^{\mathrm{T}} \bar{\boldsymbol{v}} = q(x_j),$$

since $V^{\mathrm{T}} R \bar{\boldsymbol{d}} = \{q(x_k)\}$ and $V^{\mathrm{T}} \bar{\boldsymbol{v}} = \{x_k v(x_k)\} = \{\delta_{k-j} \tilde{p}'(x_k)\}$. Hence, the sum is non-zero because $x_j \neq y_i$ for all $i, j > 0$. The same argument can be used for any j, which proves the theorem for $\{x_j\}$. The proof for $\{y_j\}$ is identical upon exchanging the roles of $\boldsymbol{c}$, $\tilde{\boldsymbol{a}}$ and $\boldsymbol{d}$, $\tilde{\boldsymbol{b}}$. This leads to (4). $\square$

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