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Algebraic Geometry

Twisted Chern classes and $\mathbb{G}_m$-gerbes

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Abstract

Using the language of stacks one can give a simple definition of functorial Chern classes for twisted sheaves. Calculating the cohomology ring of a $\mathbb{G}_m$-gerbe we observe that the twisted Chern classes used by Huybrechts and Stellari are specializations of these classes. We describe explicitly the relation between the choice of a cocycle in the definition of twisted sheaves and the 2-categorical structure of $\mathbb{G}_m$-gerbes. ***To cite this article: J. Heinloth, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Classes de Chern tordues et gerbes liées par $\mathbb{G}_m$. La théorie des champs nous permet de donner une définition simple et fonctorielle des classes de Chern des faisceaux tordus. Le calcul de l'anneau de cohomologie d'une gerbe liée par $\mathbb{G}_m$ démontre que les classes de Chern tordues, introduites par Huybrechts et Stellari, sont des spécialisations de ces classes. De plus, nous expliquons la relation entre le choix d'un cocycle utilisé dans la définition des faisceaux tordus et le fait que les gerbes forment une 2-catégorie. ***Pour citer cet article : J. Heinloth, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

1. Introduction

Recently, Huybrechts and Stellari defined cohomological Chern classes for twisted sheaves on smooth projective varieties and used these in the study of derived equivalences on K3 surfaces [5,6]. In these articles the authors also point out that the new Chern classes for twisted sheaves do not behave as functorially as one would expect. We note that these problems disappear if one uses the language of $\mathbb{G}_m$-gerbes. In particular, in this setup, Chern classes take values in the cohomology of a gerbe over a space X, which is a polynomial ring over $H^*(X,\mathbb{Q})$ if the class of the gerbe is torsion. Furthermore, the choice of a cocycle appearing in the definition of twisted sheaves in loc. cit. can be explained by the fact that gerbes form a 2-category, whose structure is given by a truncated cohomology complex instead of a cohomology group.

These results are probably well known to specialists, but since I could not find them in the literature I thought that it might be useful to give a short explanation of these facts.

In the following X is either a variety over $\mathbb{C}$ or a differentiable manifold. In the differentiable setting the results also hold if one replaces the multiplicative group $\mathbb{G}_m$ by S^1.

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2. Cohomological Chern classes for vector bundles on stacks

Let E be a vector bundle of rank r on X. It defines a map $f_E : X \to \mathrm{BGL}_r = [pt/\mathrm{GL}_r]$ to the classifying stack (or space) of GL_r-bundles. Since $H^*(\mathrm{BGL}_r, \mathbb{Z}) = \mathbb{Z}[c_1, \ldots, c_r]$ (cf. [2]) one can define cohomological Chern classes of E as $c_i(E) = f_E^* c_i$. By descent for vector bundles the same holds for any algebraic stack (in the sense of Artin) or any differentiable stack $\mathcal{M}$, i.e. any vector bundle E on $\mathcal{M}$ defines a map $\mathcal{M} \to \mathrm{BGL}_r$. To describe this map explicitly in terms of groupoids, one can choose an atlas $\pi : X \to \mathcal{M}$ such that π^*E is trivial. After trivializing, the descent datum for E is a map $X \times_{\mathcal{M}} X \to \mathrm{GL}_r$ and thus defines a map of groupoids $(X \times_{\mathcal{M}} X \rightrightarrows X) \to (\mathrm{GL}_r \rightrightarrows pt)$ which defines a map $\mathcal{M} \to \mathrm{BGL}_r$.

We can again define cohomological Chern classes $c_i(E) := f_E^*(c_i) \in H^*(\mathcal{M}, \mathbb{Z})$.

3. Cohomology of $\mathbb{G}_m$-gerbes

Given a class $\tau \in H^2(X, \mathbb{G}_m) = H^2(X, \mathcal{O}_X^*)$ we choose a $\mathbb{G}_m$-gerbe $X^\tau \to X$ over X in this class (the relation between this choice and the choice of a cocycle for τ is explained in the last section). Let $d(\tau) \in H^3(X, \mathbb{Z})$ be the Dixmier–Douady class of τ, given by the boundary of the exponential sequence.

Lemma 3.1. *Let $\pi : X^\tau \to X$ be a $\mathbb{G}_m$-gerbe over a variety and E a vector bundle of weight 1 on X^τ. Then $H^*(X^\tau, \mathbb{Q}) \cong H^*(X, \mathbb{Q})[z]$, where $z \in H^2(X^\tau, \mathbb{Q})$ is the first Chern class of E.*

The same holds in the differentiable setting if $\tau \in H^2(X, S^1)_{\mathrm{tors}}$ is a torsion element.

Recall that a vector bundle E on X^τ is of weight 1, if the $\mathbb{G}_m$-automorphisms of all points act by scalar multiplication on the fibres of E, equivalently E is a τ-twisted vector bundle on X.

Proof. The Leray spectral sequence for π: $E_2^{p,q} = H^p(X, \mathbf{R}^q\pi_*\mathbb{Q}) \Rightarrow H^{p+q}(X^\tau, \mathbb{Q})$, degenerates, since $\tau^{\leqslant c}\mathbf{R}\pi_*\mathbb{Q}$ is the same as the cohomology of a projective bundle: We define $E' := E^{\oplus(c+1)}$, which is a vector bundle of weight 1 on X^τ, $\mathrm{rk}(E') > c$ and $c_1(E') = (c+1)c_1(E)$. Denote by $s_0 : X^\tau \to E'$ the zero section of E'. Then $p : E' - s_0(X^\tau) \to X^\tau$ is $2\,\mathrm{rk}(E') - 2$-acyclic (i.e., $\mathbf{R}^i p_*\mathbb{Q} = 0$ for $0 < i < 2\,\mathrm{rk}(E') - 1$), therefore $H^*(E' - s_0(X^\tau), \mathbb{Q}) \cong H^*(X^\tau, \mathbb{Q})$ for $* < 2\,\mathrm{rk}(E') - 1$. Since E' is of weight 1, $\bar{p} : E' - s_0(X^\tau) \to X$ is a bundle of projective spaces and moreover the class $p^*(c_1(E'))$ gives a generator for the rational cohomology of the fibres of $\bar{p}$. Thus $\mathbf{R}^i\bar{p}_*\mathbb{Q}$ are constant sheaves and the spectral sequence for the cohomology of the projective bundle degenerates by the theorem of Leray–Hirsch. □

Remark 1. The class z generating the cohomology of the gerbe as algebra over $H^*(X, \mathbb{Q})$ depends on the choice of the vector bundle E of weight 1 on X^τ. For example, if we tensor E with the pull back of a line bundle L on X we change z by adding $\pi^*(c_1(L))$.

3.1. Remark on integral coefficients

The lemma does not hold for integral coefficients. It also fails in the differentiable setting, if the Dixmier–Douady class $d(\tau)$ of the gerbe is not torsion: The Leray spectral sequence still looks like:

$$\begin{array}{ccccc}
H^0(X, \mathbf{R}^2\pi_*\mathbb{Z}) & \cdots & & & \\
0 & 0 & 0 & 0 & \cdots \\
H^0(X, \mathbb{Z}) & H^1(X, \mathbb{Z}) & H^2(X, \mathbb{Z}) & H^3(X, \mathbb{Z}) & \cdots
\end{array}$$
$$d_3 : H^0(X, \mathbf{R}^2\pi_*\mathbb{Z}) \to H^3(X, \mathbb{Z})$$

where d_3 is the first differential that can be non-trivial. Since the gerbe is neutral over itself we have $\pi^*(d(\tau)) = 0$, therefore $d(\tau)$ must lie in the image of d_3. Thus $H^0(X, \mathbf{R}^2\pi_*\mathbb{Z}) \neq 0$ so that the locally constant sheaf $\mathbf{R}^2\pi_*\mathbb{Z}$ must be constant.

Since the spectral sequence is multiplicative, this shows that for twists such that $d(\tau)$ is not a torsion class, the cohomology of X^τ only contains a quotient of $H^*(X, \mathbb{Z})$.

More precisely, the differential d_3 maps a generator of $H^0(X, \mathbf{R}^2\pi_*\mathbb{Z}) \cong \mathbb{Z}$ to $d(\tau)$. This can be seen by looking at the exponential sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} \cong \pi_*\mathbb{Z} & \longrightarrow & \mathcal{O}_X \cong \pi_*\mathcal{O}_{X^\tau} & \longrightarrow & \mathcal{O}_X^* \cong \pi_*\mathcal{O}_{X^\tau}^* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{R}\pi_*\mathbb{Z} & \longrightarrow & \mathbf{R}\pi_*\mathcal{O}_{X^\tau} & \longrightarrow & \mathbf{R}\pi_*\mathcal{O}_{X^\tau}^* & \longrightarrow & 0. \end{array}$$

The sheaf $\mathbf{R}^1\pi_*\mathcal{O}_{X^\tau}^*$ is the sheafification of $U \mapsto \mathrm{Pic}(U \times_X X^\tau)$. But for acyclic open sets U we have $U \times_X X^\tau \cong U \times B\mathbb{G}_m$, so that $\mathrm{Pic}(U \times_X X^\tau) \cong \mathbb{Z} \cong H^2(U \times \mathbb{G}_m, \mathbb{Z})$ and the canonical generator is given by (the Chern class of) the universal line bundle. Thus $\mathbf{R}^1\pi_*\mathbb{G}_m \cong \mathbf{R}^2\pi_*\mathbb{Z} \cong \mathbb{Z}$. Furthermore, the differential $d_2 : H^0(X, \mathbf{R}^1\pi_*\mathcal{O}_X^*) \to H^2(X, \mathbb{G}_m)$ maps the generator to τ: Choose an acyclic covering U_i of X and universal bundles L_i on $U_i \times_X X^\tau$. Then the differential in the spectral sequence is given by the obstruction to glueing of these bundles, which equals the class τ. Applying the morphism of spectral sequences induced by the exponential establishes the claim.

4. Comparison with the twisted Chern character of Huybrechts and Stellari

In [5], Huybrechts and Stellari define a twisted Chern character for a twist $\tau \in H^2(X, \mathbb{G}_m)$ depending on the choice of a class $B \in H^2(X, \mathbb{Q})$ with $\exp(B^{0,2}) = \tau$. Twisted bundles on X are the same as bundles of weight 1 on the corresponding gerbe and their construction gives a specialization of the Chern classes defined above as follows.

First note that the exponential sequence: $H^2(X, \mathcal{O}_X) \xrightarrow{\exp} H^2(X, \mathcal{O}_X^*) \xrightarrow{d} H^3(X, \mathbb{Z})$ shows that if $\tau = \exp(B^{0,2})$ then the Dixmier–Douady class $d(\tau) = 0$. In particular, the twist τ is trivial in the differentiable category (the sheaf of differentiable functions is acyclic).

Thus there exists a differentiable line bundle L of weight 1 on X^τ (equivalently a twisted line bundle L on X). Then for any vector bundle E of weight 1 on X^τ the tensor product $E \otimes L^{-1}$ is of weight 0. Therefore $E \otimes L^{-1}$ descends to a bundle on X where one can apply the usual Chern character. Call the resulting class $\mathrm{ch}_L(E)$.

Alternatively, the choice of L defines a class $z = c_1(L) \in H^2(X^\tau, \mathbb{Q})$ and $\mathrm{ch}_L(E)$ is obtained by setting $z = 0$ in the Chern character of E on X^τ considered as a powerseries in z ($\mathrm{ch}(E) = \mathrm{ch}(E \otimes L^{-1}) \exp(z)$ and the first factor lies in $H^*(X, \mathbb{Q}) \subset H^*(X^\tau, \mathbb{Q})$).

Huybrechts and Stellari point out that a canonical choice for the Chern class of L is already determined by B: Indeed, denote by S^1_{diff} (resp. $\mathbf{R}_{\mathrm{diff}}$) the sheaf of differentiable sections with values in S^1 (resp. in $\mathbf{R}$). To define X^τ by τ we have to choose a 3-cocycle $\tau_{ijk} \in Z^3(X, \mathcal{O}_X^*)$ and since $\tau = \exp(B^{0,2})$ we can even choose $\tau_{ijk} \in Z^3(X, S^1_{\mathrm{diff}})$. This can be done through a choice of a cocycle for B in $Z^3(X, \mathbf{R})$. We have:

$$\begin{array}{ccccccc} & & & & Z^3(X, \mathbf{R}) & \longrightarrow & H^2(X, \mathbf{R}) \\ & & & & \downarrow & & \downarrow \\ Z^2(X, \mathbf{R}_{\mathrm{diff}}) & \longrightarrow & C^2(X, \mathbf{R}_{\mathrm{diff}}) & \xrightarrow{d} & Z^3(X, \mathbf{R}_{\mathrm{diff}}) & \longrightarrow & 0 \\ \downarrow{\scriptstyle\exp} & & \downarrow{\scriptstyle\exp} & & \downarrow{\scriptstyle\exp} & & \downarrow \\ Z^2(X, S^1_{\mathrm{diff}}) & \longrightarrow & C^2(X, S^1_{\mathrm{diff}}) & \xrightarrow{d} & Z^3(X, S^1_{\mathrm{diff}}) & \longrightarrow & H^2(X, S^1_{\mathrm{diff}}). \end{array}$$

Since $\mathbf{R}_{\mathrm{diff}}$ is acyclic, the cocycle for B is in fact a boundary in $Z^3(X, \mathbf{R}_{\mathrm{diff}})$. The choice of a lifting of B to an element $a \in C^2(X, \mathbf{R}_{\mathrm{diff}})$ defines (by exp) a bundle L of weight 1 on X^τ. Two such lifting differ by an element of $Z^2(X, \mathbf{R}_{\mathrm{diff}}) = d(C^1(X, \mathbf{R}_{\mathrm{diff}}))$. Thus the Chern class of the bundle L does not depend on the choice of a and B_{ijk}. Huybrechts and Stellari define the Chern class by $\mathrm{ch}_B(E) := \mathrm{ch}_L(E)$.

Remark 2. The Chern character defined on the gerbe X^τ has the advantage that it is compatible with morphisms of gerbes and does not depend on additional choices. Our approach compares to the one of Huybrechts and Stellari as follows: $\tau \in H^2(X, \mathbb{G}_m)$ corresponds to an isomorphism class of gerbes. The choice of the cocycle incorporates the choice of a gerbe in this isomorphism class (see Section 5). Note that in the applications the gerbe is usually canonical, typically defined by some moduli problem. Finally to get Chern classes with values in $H^*(X, \mathbb{Q})$, the choice of a B-field corresponds to the choice of a vector bundle of weight 1.

5. Choosing cocycles and the 2-category of gerbes

It is well-known that the set of isomorphism classes of $\mathbb{G}_m$-gerbes on X is naturally isomorphic to $H^2(X, \mathbb{G}_m)$ (e.g., [3,1]). Stacks form a 2-category, and there is no canonical choice of such an isomorphism class. Thus, to define twisted sheaves on X one usually chooses a cocycle for a given cohomology class $\tau \in H^2(X, \mathbb{G}_m)$. This choice also determines a gerbe and the relation to the 2-categorical structure can be explained by recalling the proof of the result quoted above (the arguments work for any Abelian group instead of $\mathbb{G}_m$):

Let $0 \to A_0 \to A_1 \to A_2 \to 0$ be a complex of Abelian groups representing the complex $\tau_{\leqslant 2}\mathbf{R}\Gamma(X, \mathbb{G}_m)$.

In the same way as a two term complex $A_0 \to A_1$ defines a groupoid (just because A_0 acts on A_1) one can define a 2-category from the above 3-term complex: The objects are elements in A_2; an object in $\mathrm{Hom}(x, y)$ is an element $\phi \in A_1$ with $x + d(\phi) = y$ and a morphism $\Phi : \phi \to \psi$ is an element $\Phi \in A_0$ with $\phi + d(\Phi) = \psi$.

Proposition 5.1. *The* 2*-category of* $\mathbb{G}_m$*-gerbes is equivalent to the two category defined by any* 3*-term complex representing* $\tau_{\leqslant 2}\mathbf{R}\Gamma(X, \mathbb{G}_m)$.

For 2-categories and equivalences between them see [4], Section 1. The proof below is the standard proof [1], keeping track of the morphisms.

Proof. First, we use a particular complex $A_\bullet$ as above and construct:

- For any element a_2 in A_2 a gerbe $X^{a_2} \to X$.
- For any $a_2 \in A_2$ and any element $a_1 \in A_1$ a morphism $X^{a_2} \to X^{a_2+d(a_1)}$ such that this defines an equivalence of categories: $\langle A_0 \to \ker(A_1 \to A_2)\rangle \to \mathrm{Isom}_{\mathbb{G}_m\text{-gerbes}/X}(X^{a_2}, X^{a_2+d(a_1)})$.

To construct a resolution of the sheaf $\mathbb{G}_m$ we choose a contractible covering U_i of X such that all intersections $U_{i_1,\ldots,i_r} = U_{i_1} \cap \ldots \cap U_{i_r}$ are also acyclic. We calculate the cohomology of X by the Cech complex: $\oplus\Gamma(U_i, \mathbb{G}_m) \to \oplus\Gamma(U_{ij}, \mathbb{G}_m) \to \oplus\Gamma(U_{ijk}, \mathbb{G}_m) \to \oplus\Gamma(U_{ijkl}, \mathbb{G}_m)$.

Let a_{ijk} be a cocycle. Define a groupoid: $\coprod \mathbb{G}_m \times U_{ij} \rightrightarrows \coprod U_i$, where the source and target morphisms are given by the projection to U_i and U_j and the composition $m : (\mathbb{G}_m \times U_{ij}) \times_{U_j} (\mathbb{G}_m \times U_{jk}) \to \mathbb{G}_m \times U_{ik}$ is defined as: $m(s_{ij}, x, s_{jk}, x) := (s_{ij} \cdot s_{jk} \cdot a_{ijk}, x)$. This composition is associative: we drop the point $x \in X$ from the notation and take four indices i, j, k, l and sections s_{ij}, s_{jk}, s_{kl}. Then $m(m(s_{ij}, s_{jk}), s_{kl}) = s_{ij}s_{jk}s_{kl}a_{ijk}a_{ikl}$ and $m(s_{ij}, m(s_{jk}, s_{kl})) = s_{ij}s_{jk}s_{kl}a_{jkl}a_{ijl}$. Since a_{ijk} is a cocycle, we know that $a_{ijk}a_{ijl}^{-1}a_{ikl}a_{jkl}^{-1} = 0$ and the two compositions coincide.

The gerbe $X^{a_{ijk}}$ over X is the stack defined by this groupoid:

$$\begin{array}{ccccc} \coprod \mathbb{G}_m \times U_{ij} & \rightrightarrows & \coprod U_i & \longrightarrow & X^{a_{ijk}} \\ \downarrow & & \downarrow & & \downarrow \\ U_{ij} & \rightrightarrows & \coprod U_i & \longrightarrow & X. \end{array}$$

Furthermore, given $b_{ij} \in \coprod \Gamma(U_{ij}, \mathbb{G}_m)$ we can define a map of groupoids by: $(s_{ij}, x) \mapsto (b_{ij}s_{ij}, x)$. This is compatible with m since:

$$(s_{ij}b_{ij})(s_{jk}b_{jk})(a_{ijk}b_{ij}^{-1}b_{ik}b_{jk}^{-1}) = (s_{ij}s_{jk}a_{ijk})b_{ik}.$$

Finally, an element $c_i \in \coprod \Gamma(U_i, \mathbb{G}_m)$ defines a 2-morphism: $U_{ij} \to \mathbb{G}_m \times U_{ij}$ by $x \mapsto c_i c_j^{-1}$.

It is easy to check that these maps define the claimed equivalences, since any $\mathbb{G}_m$-gerbe restricted to the contractible space U_i is trivial and automorphisms of the trivial gerbe are given by line bundles on the base. Thus the choice of trivializations of the restrictions of the gerbe to U_i and a choice of trivialisations of the line bundles obtained from the two trivialisations on U_{ij} defines a cocycle a_{ijk}. Similarly one checks the claim on morphisms.

It is immediate from the definitions that quasi-isomorphic complexes $A_\bullet \to B_\bullet$ define equivalent 2-categories. □

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