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C. R. Acad. Sci. Paris, Ser. I 341 (2005) 583–586
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Probability Theory

The lattice-theoretic structure of sets of bivariate copulas and quasi-copulas

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Received 20 July 2005; accepted after revision 15 September 2005

Available online 11 October 2005

Presented by Paul Deheuvels

Abstract

In this Note we show that the set of quasi-copulas is a complete lattice, which is order-isomorphic to the Dedekind–MacNeille completion of the set of copulas. Consequently, any set of copulas sharing a particular statistical property is guaranteed to have pointwise best-possible bounds within the set of quasi-copulas. ***To cite this article: R.B. Nelsen, M. Úbeda Flores, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

La structure réseau-théorique des ensembles de copules et quasi-copules bivariées. Dans cette Note, nous montrons que l'ensemble des quasi-copules est un treillis complet, qui est isomorphe au sens de l'ordre à la complétion de Dedekind–MacNeille de l'ensemble des copules. En conséquence, tout ensemble de copules qui possède une propriété statistique particulière est assuré de réaliser les meilleures bornes ponctuelles parmi l'ensemble des quasi-copules. ***Pour citer cet article : R.B. Nelsen, M. Úbeda Flores, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

1. Introduction

Copulas – bivariate distribution functions with uniform margins – have proven to be remarkably useful in statistical modelling and in the study of dependence and association of random variables. Quasi-copulas, a more general concept, share many properties with copulas. The set of copulas is a proper subset of the set of quasi-copulas, and both sets have a natural partial ordering. The purpose of this Note is to investigate some properties of those partially ordered sets (posets).

A *copula* is a function $C:[0,1]^2 \to [0,1]$ which satisfies (C1) the boundary conditions $C(t,0) = C(0,t) = 0$ and $C(t,1) = C(1,t) = t$ for all $t \in [0,1]$, and (C2) the 2-increasing property, i.e., $V_C([u_1,u_2] \times [v_1,v_2]) = C(u_2,v_2) -$

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2005.09.026

$C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geqslant 0$ for all u_1, u_2, v_1, v_2 in $[0, 1]$ such that $u_1 \leqslant u_2$ and $v_1 \leqslant v_2$. The importance of copulas in statistics stems in part from Sklar's theorem [6]: Let H be a bivariate distribution function with margins F and G. Then there exists a copula C (which is uniquely determined on Range $F \times$ Range G) such that $H(x, y) = C(F(x), G(y))$ for all x, y in $[-\infty, \infty]$. Thus copulas link joint distribution functions to their margins. For any copula C we have $W(u, v) = \max(0, u + v - 1) \leqslant C(u, v) \leqslant \min(u, v) = M(u, v)$ for all (u, v) in $[0, 1]^2$. M and W are copulas, and the order relation in the above inequality leads to a partial order $\prec$ (also known as *concordance order*) on the set $\mathbf{C}$ of copulas: $C_1 \prec C_2$ if and only if $C_1(u, v) \leqslant C_2(u, v)$ for all (u, v) in $[0, 1]^2$. See [4] for more details.

The concept of a quasi-copula was introduced by Alsina et al. [1] in order to characterize operations on distribution functions that can or cannot be derived from operations on random variables defined on the same probability space. A *quasi-copula* is a function $Q : [0, 1]^2 \to [0, 1]$ which satisfies condition (C1), but in place of (C2), the weaker conditions (i) Q is non-decreasing in each variable, and (ii) the Lipschitz condition $|Q(u_1, v_1) - Q(u_2, v_2)| \leqslant |u_1 - u_2| + |v_1 - v_2|$ for all (u_1, v_1), (u_2, v_2) in $[0, 1]^2$ (see [3]). While every copula is a quasi-copula, there exist *proper* quasi-copulas, i.e., quasi-copulas which are not copulas. As with copulas, the set $\mathbf{Q}$ of quasi-copulas is also partially ordered by $\prec$, and for any quasi-copula Q we have $W \prec Q \prec M$. Finally, $\mathbf{Q} \setminus \mathbf{C}$ denotes the set of proper quasi-copulas.

We will also need some notions from lattice theory. Given two elements x and y of a poset $(P, \prec)$, let $x \vee y$ denote the *join* of x and y (when it exists); similarly for $\bigvee S$, where S is a subset of P; $x \wedge y$ denotes the *meet* of x and y (when it exists); and similarly for $\bigwedge S$. In particular, for any pair Q_1 and Q_2 of quasi-copulas (or copulas), $Q_1 \vee Q_2 = \inf\{Q \in \mathbf{Q} \mid Q_1 \prec Q,\ Q_2 \prec Q\}$ and $Q_1 \wedge Q_2 = \sup\{Q \in \mathbf{Q} \mid Q \prec Q_1,\ Q \prec Q_2\}$. If the join or meet is found within a particular poset P, we subscript $\bigvee_P S$. Given two posets A and B, we say that A is *join-dense* (respectively, *meet-dense*) in B if for any D in B, there exists a set $S \subseteq A$ such that $D = \bigvee S$ (respectively, $D = \bigwedge S$). If $x \in P$, then $\downarrow x = \{s \in P \mid s \prec x\}$ and $\uparrow x = \{s \in P \mid s \succ x\}$. A poset $P \neq \emptyset$ is a *lattice* if for every x, y in P, $x \vee y$ and $x \wedge y$ are in P; and P is a *complete lattice* if for every $S \subseteq P$, $\bigvee S$ and $\bigwedge S$ are in P.

2. The lattice of quasi-copulas

We begin with some basic results on the structure of the posets $\mathbf{Q}$, $\mathbf{C}$ and $\mathbf{Q} \setminus \mathbf{C}$.

Theorem 2.1. $\mathbf{Q}$ *is a complete lattice; however, neither* $\mathbf{C}$ *nor* $\mathbf{Q} \setminus \mathbf{C}$ *is a lattice.*

Proof. Let S be any set of quasi-copulas, and define $\overline{Q}_S(u, v) = \sup\{Q(u, v) \mid Q \in S\}$ and $\underline{Q}_S(u, v) = \inf\{Q(u, v) \mid Q \in S\}$ for each (u, v) in $[0, 1]^2$. Since $\overline{Q}_S$ and $\underline{Q}_S$ are quasi-copulas [5, Theorem 2.2], it now follows that $\bigvee S$ $(= \overline{Q}_S)$ and $\bigwedge S$ $(= \underline{Q}_S)$ are in $\mathbf{Q}$, hence $\mathbf{Q}$ is a complete lattice.

Now suppose that $\mathbf{C}$ is a lattice, and consider the following copulas: $C_1(u, v) = \min(u, v, \max(0, u - 2/3, v - 1/3, u + v - 1))$, $C_2(u, v) = C_1(v, u)$, $C_3(u, v) = \min(u, v, \max(0, u - 1/3, v - 1/3, u + v - 2/3))$ and $C_4(u, v) = \min(u, v, \max(1/3, u - 1/3, v - 1/3, u + v - 1))$. The copulas $C_1, \ldots, C_4$ are singular, and the support of each one consists of two or three line segments in $[0, 1]^2$ with slope $+1$, as shown in Fig. 1. If $\mathbf{C}$ is a lattice, $C = C_1 \vee C_2$ exists and is a copula. Hence $C(1/3, 2/3) \geqslant C_1(1/3, 2/3) = 1/3 = M(1/3, 2/3)$, so that $C(1/3, 2/3) = 1/3$. Similarly (using C_2), $C(2/3, 1/3) = 1/3$. Since $C_1 \prec C_3$ and $C_2 \prec C_3$, $C \prec C_3$ and so $C(1/3, 1/3) \leqslant C_3(1/3, 1/3) = 0$, thus $C(1/3, 1/3) = 0$. Similarly $C(2/3, 2/3) \leqslant C_4(2/3, 2/3) = 1/3 = W(2/3, 2/3)$, so $C(2/3, 2/3) = 1/3$. Hence $V_C([1/3, 2/3]^2) = -1/3$, i.e., C is a proper quasi-copula; a contradiction.

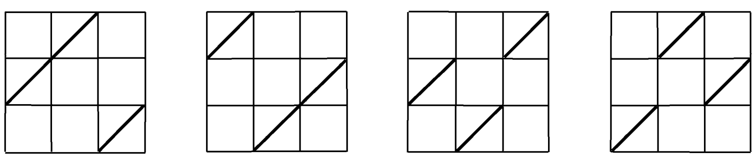

Fig. 1. The supports of C_1, C_2, C_3, and C_4 (left to right).

To prove that $\mathbf{Q} \setminus \mathbf{C}$ is not a lattice, it suffices to exhibit two proper quasi-copulas Q_1 and Q_2 whose join (or meet) is a copula. Let Q be the proper quasi-copula $C_1 \vee C_2$ above, and define

$$Q_1(u,v) = \begin{cases} (1/2)Q(2u,2v), & (u,v) \in B_1, \\ M(u,v), & \text{elsewhere}, \end{cases} \quad \text{and} \quad Q_2(u,v) = \begin{cases} (1/2)(1+Q(2u-1,2v-1)), & (u,v) \in B_2, \\ M(u,v), & \text{elsewhere}, \end{cases}$$

where $B_1 = [0,1/2]^2$ and $B_2 = [1/2,1]^2$. It is easy to verify that Q_1 and Q_2 are quasi-copulas, and that $Q_1 \vee Q_2 = M$, which is a copula rather than a proper quasi-copula. □

Lemma 2.2. *Let $(a,b) \in (0,1)^2$, let $\theta \in [W(a,b), M(a,b)]$, and define $S_{(a,b),\theta} = \{Q \in \mathbf{Q} \mid Q(a,b) = \theta\}$. Then $\bigvee S_{(a,b),\theta}$ and $\bigwedge S_{(a,b),\theta}$ are the copulas given by $\bigvee S_{(a,b),\theta}(u,v) = \min(M(u,v), \theta + (u-a)^+ + (v-b)^+)$ and $\bigwedge S_{(a,b),\theta}(u,v) = \max(W(u,v), \theta - (a-u)^+ - (b-v)^+)$, where $x^+ = \max(x,0)$.*

Proof. Let Q be any quasi-copula. The defining conditions for quasi-copulas (nondecreasing and Lipschitz in each variable) yield, for all $(u,v) \in [0,1]^2$, the inequalities $-(a-u)^+ \leqslant Q(u,v) - Q(a,v) \leqslant (u-a)^+$ and $-(b-v)^+ \leqslant Q(a,v) - Q(a,b) \leqslant (v-b)^+$, hence $\theta - (a-u)^+ - (b-v)^+ \leqslant Q(u,v) \leqslant \theta + (u-a)^+ + (v-b)^+$. Thus $\bigwedge S_{(a,b),\theta} \prec Q \prec \bigvee S_{(a,b),\theta}$, and these bounds are copulas [4, Theorem 3.2.2]. □

Lemma 2.3. *Let $Q \in \mathbf{Q}$ be any quasi-copula, and let $S = (\downarrow Q)_{\mathbf{C}} = \{C \in \mathbf{C} \mid C \prec Q\}$. Then $\bigvee_{\mathbf{Q}} S = Q$.*

Proof. Let (a,b) any point in $(0,1)^2$, and set $\theta = Q(a,b)$. From Lemma 2.2, $\bigwedge S_{(a,b),\theta} \in S$, furthermore $\bigwedge S_{(a,b),\theta}(a,b) = \theta = Q(a,b)$. Hence $\sup\{C(a,b) \mid C \in S\} = Q(a,b)$. □

Note that Lemma 2.3 also holds with $S = (\uparrow Q)_{\mathbf{C}} = \{C \in \mathbf{C} \mid C \succ Q\}$, so that $\bigwedge_{\mathbf{Q}} S = Q$.

As a consequence of Lemma 2.3 and the definitions of join-dense and meet-dense, we have

Lemma 2.4. ***C** is join-dense and meet-dense in **Q**.*

Before proving the main result in this section, we need several more lattice-theoretic concepts and results. Let S be a subset of a poset $(P,\prec)$. The set S^u of upper bounds of S is given by $S^u = \{x \in P \mid \forall s \in S, s \prec x\}$; and similarly $S^l = \{y \in P \mid \forall s \in S, s \succ y\}$ denotes the set of lower bounds of S. Also note that if $x \in P$, then $(\downarrow x)^u = \uparrow x$ and $(\uparrow x)^l = \downarrow x$. If $\varphi : P \to L$ is an *order-imbedding* (i.e., order-preserving injection) of a poset P into a complete lattice L, then we say that L is a *completion* of P. The *Dedekind–MacNeille completion* (or *normal completion*, or *completion by cuts*) of a poset P is given by $\mathrm{DM}(P) = \{A \subseteq P \mid (A^u)^l = A\}$ (which, ordered by $\subseteq$, is a complete lattice). The order-imbedding φ above is given by $\varphi(x) = \downarrow x = ((\downarrow x)^u)^l \in \mathrm{DM}(P)$. Finally, if φ maps P *onto* L, φ is an *order-isomorphism* (i.e., order-preserving bijection).

Theorem 2.5. ***Q** is order-isomorphic to the Dedekind–MacNeille completion of **C**.*

Proof. This is a consequence [2, Theorem 7.41] of the fact that $\mathbf{C}$ is both join-dense and meet-dense in $\mathbf{Q}$. The order-isomorphism $\varphi : \mathbf{Q} \to \mathrm{DM}(\mathbf{C})$ is given by $\varphi(Q) = (\downarrow Q)_{\mathbf{C}}$. □

Thus the set of quasi-copulas is a lattice-theoretic completion of the set of copulas, analogous to Dedekind's construction of the reals as a completion by cuts of the set of rationals. Consequently, we can give the following characterization of quasi-copulas in terms of copulas, based on the order-isomorphism in Theorem 2.5.

Corollary 2.6. *Let $Q : [0,1]^2 \to [0,1]$. Then Q is a quasi-copula if and only if there exists a set S of copulas such that $Q = \vee_{\mathbf{Q}} S$.*

Proof. Let Q be a quasi-copula, and let $S = (\downarrow Q)_{\mathbf{C}}$. Since $W \prec Q$ and $W \in \mathbf{C}$, we have $S \neq \emptyset$. Then by Lemma 2.3, $Q = \bigvee_{\mathbf{Q}} S$. Conversely, let $f : [0,1]^2 \to [0,1]$ for which there exists a set S of copulas such that $f = \bigvee_{\mathbf{Q}} S$. Then f is a quasi-copula, since $\mathbf{Q}$ is complete. □

Corollary 2.6 also holds with joins replaced by meets.

In the proof of Theorem 2.1 we used quasi-copulas which were the join of a finite number (two) of copulas. However, there exist quasi-copulas which cannot be written as the meet or join of any finite set of copulas. The following result proves the result for meets (joins are similar).

Proposition 2.7. *Let Q be a quasi-copula for which $Q(u,v) = \max(u - 1/3, v - 1/3)$, $(u,v) \in [1/3, 2/3]^2$, and let $\mathbf{C}_0$ denote any set of copulas such that $Q = \bigwedge \mathbf{C}_0$. Then $\mathbf{C}_0$ has infinitely many members.*

Proof. We first note that there exist quasi-copulas Q with the property $Q(u,v) = \max(u - 1/3, v - 1/3)$ for $(u,v) \in [1/3, 2/3]^2$ [5, Example 2.1]. Let $\mathbf{C}_0$ be any set of copulas such that $Q = \bigwedge \mathbf{C}_0$, and let C be a (fixed) element of $\mathbf{C}_0$. Since $Q(1/3, 2/3) = 1/3 = M(1/3, 2/3)$, it follows that $C(1/3, 2/3) = 1/3$; and similarly $C(2/3, 1/3) = 1/3$. Thus for some ε, δ in $[0, 1/3]$ with $\varepsilon + \delta \geqslant 1/3$, $C(1/3, 1/3) = \varepsilon$ and $C(2/3, 2/3) = 1/3 + \delta$. Now let (u,v) be a (fixed) point in $[1/3, 2/3]^2$. Then $V_C([u,1] \times [v, 2/3]) \geqslant 0$ implies $C(u,v) \geqslant C(u, 2/3) + v - 2/3 \geqslant v - 1/3$, and similarly $C(u,v) \geqslant u - 1/3$. Furthermore, $V_C([u,1] \times [v,1]) \geqslant \delta$ implies $C(u,v) \geqslant u + v - 1 + \delta$, and hence $C(u,v) \geqslant \max(\varepsilon, u - 1/3, v - 1/3, u + v - 1 + \delta)$ for any (u,v) in $[1/3, 2/3]^2$. But $\max(\varepsilon, u - 1/3, v - 1/3, u + v - 1 + \delta) = v - 1/3$ only on the rectangle $[1/3, 2/3 - \delta] \times [1/3 + \varepsilon, 2/3]$, a proper subset of the triangle $\{(u,v) \mid 1/3 \leqslant u \leqslant v \leqslant 2/3\}$ where $Q(u,v) = v - 1/3$, and hence Q cannot be the meet of a finite number of copulas. $\square$

Acknowledgements

The authors acknowledge the support of the Junta de Andalucía (Spain), and their respective institutions. The second author also thanks the support by the Ministerio de Ciencia y Tecnología (Spain) under research project BFM2003-06522.

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