



Group Theory

Explicitation of characters of the symmetric group

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Abstract

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Résumé

Explicitation des caractères du groupe symétrique. Nous donnons une formule totalement explicite pour les caractères normés du groupe symétrique. *Pour citer cet article : M. Lassalle, C. R. Acad. Sci. Paris, Ser. I 341 (2005).*

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Version française abrégée

Les caractères des représentations irréductibles du groupe symétrique jouent un rôle important dans de nombreux domaines des mathématiques. Cependant depuis les premiers travaux de Frobenius [2] en 1900, on n'en connaissait aucune expression générale explicite. Seuls quelques rares cas particuliers avaient été calculés [2,4,11]. Le but de cette Note est de présenter la solution de ce problème.

Les représentations irréductibles du groupe symétrique S_n de n lettres sont indicées par les partitions λ de n . Leurs caractères χ^λ sont évalués sur les classes de conjugaison de S_n , chacune étant définie par une partition μ qui donne son type cyclique. Soit χ_μ^λ la valeur du caractère χ^λ pris sur une permutation de type cyclique μ .

Dans cette Note nous donnons une formule, générale et totalement explicite, pour le caractère normé $\hat{\chi}_\mu^\lambda = \chi_\mu^\lambda / \dim \lambda$. Cette expression est obtenue en écrivant ce caractère comme un polynôme symétrique en les « contenus » de la partition λ .

Ce polynôme symétrique est lui même défini au moyen de la nouvelle famille de nombres entiers que nous avons introduite dans [8]. Ces nombres entiers apparaissent par le calcul. Leur rapport avec la structure du groupe symétrique reste à clarifier.

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1. Introduction

The characters of the irreducible representations of the symmetric group play an important role in many areas of mathematics. However, since the early work of Frobenius [2] in 1900, no explicit formula was found for them. Explicit formulas were only computed for very few examples [2,4,11]. The purpose of this Note is to give a general, totally explicit, expression for these characters.

The irreducible representations of the symmetric group S_n of n letters are labelled by partitions λ of n (i.e. weakly decreasing sequences of positive integers summing to n). Their characters χ^λ are evaluated at a conjugacy class of S_n , labelled by a partition μ giving the cycle-type of the class. Let χ_{μ}^{λ} be the value of the character χ^λ at a permutation of cycle-type μ . We shall give an explicit formula for the normalized character $\hat{\chi}_{\mu}^{\lambda} = \chi_{\mu}^{\lambda} / \dim \lambda$.

It should be first emphasized that our result gives the dependence of $\hat{\chi}_{\mu}^{\lambda}$ with respect to λ in terms of the ‘contents’ of this partition. More precisely this character is expressed as a symmetric function evaluated on the alphabet of the contents of λ .

Such a fact was noticed by Lascoux and Thibon in 1990 (unpublished). Later it was independently conjectured by Katriel [5] and the author [7, Sections 8–11].¹

The fact that $\hat{\chi}_{\mu}^{\lambda}$ is a symmetric function in the contents of λ was recently proved in [1]. However this symmetric function remained quite obscure, even in the very elementary situation corresponding to a class μ with one non unary cycle [3]. The purpose of this note is to give it explicitly.

It is a second remarkable fact that this explicitation can only be written by using a new family of positive integers, which we have introduced in [8]. The connection of these integers with the structure of the symmetric group certainly needs to be investigated.

Detailed proofs will be given in a forthcoming paper.

2. Preliminaries

2.1. Positive integers

Let n, p, k be three integers with $0 \leq p \leq n$ and $k \geq 1$. Define

$$\binom{n}{p}_k = \frac{n}{k} \sum_{r \geq 0} \binom{p}{r} \binom{n-p}{r} \binom{n-r-1}{k-r-1}.$$

We have obviously

$$\binom{n}{p}_k = 0 \quad \text{for } k > n, \quad \binom{n}{p}_1 = n, \quad \binom{n}{p}_k = \binom{n}{n-p}_k.$$

These numbers generalize the classical binomial coefficients, since we have

$$\binom{n}{0}_k = \binom{n}{k}, \quad \binom{n}{1}_k = k \binom{n}{k}, \quad \binom{n}{p}_n = \binom{n}{p},$$

the last property being a direct consequence of the classical Chu–Vandermonde formula.

The numbers $\binom{n}{p}_k$ were studied in [8]. It was proved that they are *positive integers*, and their generating function

$$G_n(y, z) = \sum_{p=0}^n \sum_{k=1}^n \binom{n}{p}_k y^p z^k$$

was shown to be

$$G_n(y, z) = 2^{-n} \left((1+y)(1+z) + \sqrt{(1+y)^2(1+z)^2 - 4y(1+z)} \right)^n + 2^{-n} \left((1+y)(1+z) - \sqrt{(1+y)^2(1+z)^2 - 4y(1+z)} \right)^n - 1 - y^n.$$

¹ The conjectures of [7] are more general since they are given in the framework of Jack polynomials. Once specialized to $\alpha = 1$, they correspond to the characters of the symmetric group, as emphasized in Section 11 of [7].

2.2. Extension to partitions

A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is a finite weakly decreasing sequence of nonnegative integers, called parts. The number $l(\lambda)$ of positive parts is called the length of λ , and $|\lambda| = \sum_{i=1}^{l(\lambda)} \lambda_i$ the weight of λ . For any integer $i \geq 1$, $m_i(\lambda) = \text{card}\{j: \lambda_j = i\}$ is the multiplicity of the part i in λ . Clearly $l(\lambda) = \sum_{i \geq 1} m_i(\lambda)$ and $|\lambda| = \sum_{i \geq 1} i m_i(\lambda)$. We shall also write $\lambda = (1^{m_1}, 2^{m_2}, 3^{m_3}, \dots)$ and $z_\lambda = \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!$.

For any integers $0 \leq p \leq |\lambda|$ and $k \geq 1$, we define

$$\left\langle \lambda \right\rangle_p^k = \sum_{(p_i)} \sum_{(k_i)} \prod_{i=1}^{l(\lambda)} \binom{\lambda_i}{p_i}_{k_i},$$

the sum being taken over all decompositions $p = \sum_{i=1}^{l(\lambda)} p_i$, $k = \sum_{i=1}^{l(\lambda)} k_i$ with $0 \leq p_i \leq \lambda_i$ and $k_i \neq 0$ for any i . Observe that there is no such restriction for p_i .

This definition yields easily $\left\langle \lambda \right\rangle_p^k = 0$ except if $l(\lambda) \leq k \leq |\lambda|$. Indeed it is obvious that $\left\langle \lambda \right\rangle_p^k = 0$ for $k < l(\lambda)$, and since $\binom{n}{p} = 0$ for $k > n$, we have also $\left\langle \lambda \right\rangle_p^k = 0$ for $k > |\lambda|$.

For instance $\left\langle \lambda \right\rangle_1^k = 0$ except if λ is a row partition (n) , in which case $\left\langle \lambda \right\rangle_1^k = \binom{n}{p}^k$. We have easily

$$\left\langle 1 \right\rangle_k^k = k \left\langle \lambda \right\rangle_0^k, \quad \left\langle \lambda \right\rangle_p^k = \left\langle \lambda \right\rangle_{|\lambda|-p}^k, \quad \left\langle \lambda \right\rangle_{|\lambda|}^k = \binom{|\lambda|}{p}^k.$$

As a direct consequence of their definition, the generating function for the positive integers $\left\langle \lambda \right\rangle_p^k$ is the following

$$\sum_{p=0}^{|\lambda|} \sum_{k=l(\lambda)}^{|\lambda|} \left\langle \lambda \right\rangle_p^k y^p z^k = \prod_{i=1}^{l(\lambda)} G_{\lambda_i}(y, z) = \prod_{i \geq 1} (G_i(y, z))^{m_i(\lambda)}.$$

2.3. Power-sums of contents

Each partition λ may be identified with its Ferrers diagram $\{(i, j): 1 \leq i \leq l(\lambda), 1 \leq j \leq \lambda_i\}$. The content of any cell $(i, j) \in \lambda$ is defined by $c_{ij} = j - i$. Let $p_0(\lambda) = |\lambda|$ and for any integer $k > 0$, define

$$p_k(\lambda) = \sum_{(i,j) \in \lambda} (j - i)^k = \sum_{i=1}^{l(\lambda)} \sum_{j=1}^{\lambda_i} (j - i)^k.$$

Recall [9, p. 23] that the k th power-sum \mathbf{p}_k evaluated on an alphabet $A = \{a_1, a_2, a_3, \dots\}$, is given by $\mathbf{p}_k(A) = \sum_i a_i^k$. Hence $p_k(\lambda) = \mathbf{p}_k(A_\lambda)$, where $A_\lambda = \{j - i, (i, j) \in \lambda\}$ is the alphabet of the contents of λ . For any partition μ we write

$$p_\mu(\lambda) = \prod_{i=1}^{l(\mu)} p_{\mu_i}(\lambda) = \prod_{i \geq 1} p_i(\lambda)^{m_i(\mu)}.$$

For any integers $n \geq 1, k \geq 1$ and $0 \leq p \leq n$, we define

$$F_{npk}(\lambda) = \sum_{|\mu|=n} \frac{\left\langle \mu \right\rangle_p^k}{z_\mu} p_\mu(\lambda).$$

Since $\left\langle \mu \right\rangle_p^k = 0$ for $k < l(\mu)$, this sum is restricted to partitions with $l(\mu) \leq k$. Similarly since $\left\langle \mu \right\rangle_p^k = 0$ for $k > |\mu|$, one has $F_{npk}(\lambda) = 0$ for $k > n$.

For $k = 0$ the previous definition is extended by the convention $F_{np0}(\lambda) = 0$ with the only exception $F_{000}(\lambda) = 1$. For $k = 1$ we have $F_{np1}(\lambda) = p_n(\lambda)$, and for $k = n$ we obtain

$$F_{npn}(\lambda) = \binom{n}{p} \sum_{|\mu|=n} \frac{p_\mu(\lambda)}{z_\mu} = \binom{n}{p} F_{n0n}(\lambda).$$

Observe [9, p. 25] that $F_{n0n}(\lambda) = \mathbf{h}_n(A_\lambda)$, with \mathbf{h}_n the n th complete symmetric function. Finally we have $F_{n1k}(\lambda) = kF_{n0k}(\lambda)$ and $F_{npk}(\lambda) = F_{n,n-p,k}(\lambda)$.

Here is our basic tool. For any positive integer r and any indeterminate x , we write

$$c_r^\lambda(x) = \sum_{\substack{m,p,q \geq 0 \\ 2m+p+q \leq r}} (-x)^{m-1} (x+1)^p \binom{m+p+q-1}{p} \left(\sum_{k=0}^{\min(m,r-2m-p)} \binom{|\lambda|+m-1}{m-k} F_{r-2m-p,q,k}(\lambda) \right).$$

One has $c_0^\lambda(x) = -1/x$ since $F_{000}(\lambda) = 1$, and $c_1^\lambda(x) = 0$ since $F_{1q0}(\lambda) = 0$. First values are given by

$$\begin{aligned} c_2^\lambda(x) &= |\lambda|, & c_3^\lambda(x) &= 2p_1(\lambda) + (x+1)|\lambda|, \\ c_4^\lambda(x) &= 3p_2(\lambda) + 3(x+1)p_1(\lambda) - x \binom{|\lambda|+1}{2} + (x+1)^2|\lambda|. \end{aligned}$$

Finally let us mention that, by an easy argument of [6,10], each function $p_k(\lambda)$ defines a ‘shifted symmetric function’ of λ , i.e. a polynomial in the shifted variables $\lambda_i - i$. Hence the same property holds for $c_r^\lambda(x)$.

3. The one-cycle case

We briefly recall some basic notions about the characters of the symmetric group, referring the reader to [9, Section 1.7] for a more complete account.

Each permutation $\sigma \in S_n$ factorizes uniquely as a product of disjoint cycles, whose respective lengths are ordered such as to form a partition $\mu = (p_1, \dots, p_k)$. This partition determines σ up to conjugacy in S_n , and conjugacy classes are labelled by partitions μ of n . Since the irreducible representations of S_n are also labelled by partitions λ of n , we must compute χ_μ^λ , the value of the character $\chi^\lambda(\sigma)$ at a permutation σ of cycle-type μ .

We may alternatively compute the normalized character $\hat{\chi}_\mu^\lambda = \chi_\mu^\lambda / \dim \lambda$, since the dimension $\dim \lambda = \chi_{1^n}^\lambda$ of the representation λ is well known, see [9, Example 1.7.6, p. 116]. We have

$$\dim \lambda = \frac{n!}{\prod_{i=1}^n (\lambda_i + n - i)!} \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j + j - i).$$

The most elementary case corresponds to a permutation σ with one cycle of length p and all other cycles unary, associated with the partition $\mu = (p, 1^{n-p})$. As early as 1900, Frobenius [2] gave the following expression for $\hat{\chi}_\mu^\lambda$ [9, Example 1.7.7, p. 117]

$$\frac{n!}{(n-p)!} \hat{\chi}_\mu^\lambda = \sum_{i=1}^n \frac{(\lambda_i + n - i)!}{(\lambda_i + n - i - p)!} \prod_{\substack{1 \leq i, j \leq n \\ i \neq j}} \frac{\lambda_i - \lambda_j + j - i - p}{\lambda_i - \lambda_j + j - i}.$$

We shall write this quantity in a more explicit form, making obvious that it is not a rational function, but a polynomial in the variables $\lambda_i - i$.

For any indeterminate x and positive integer r , define the lowering factorial $[x]_r$ by

$$[x]_r = x(x-1) \cdots (x-r+1) = \sum_{i \geq 1} s(r, i) x^i,$$

which is the generating function for the Stirling numbers of the first kind $s(r, i)$.

Theorem 3.1. For $\mu = (p, 1^{n-p})$ we have

$$\frac{n!}{(n-p)!} \hat{\chi}_\mu^\lambda = \sum_{i \geq 2} s(p+1, i) c_i^\lambda(p).$$

For $p = 1$ the sum is restricted to $i = 2$ and we recover $\hat{\chi}_{1^n}^\lambda = 1$. Frobenius [2] computed the cases $2 \leq p \leq 4$, and Ingram [4] the case $p = 5$. As an example, for $\mu = (7, 1^{n-7})$ we have

$$\frac{n!}{(n-7)!} \hat{\chi}_\mu^\lambda = c_8^\lambda(7) - 28c_7^\lambda(7) + 322c_6^\lambda(7) - 1960c_5^\lambda(7) + 6769c_4^\lambda(7) - 13132c_3^\lambda(7) + 13068c_2^\lambda(7).$$

4. The two-cycle case

A permutation having two cycles with lengths $p \geq q$ corresponds to the partition $\mu = (p, q, 1^{n-p-q})$. Ingram [4] only computed the cases $2 \leq p \leq 4, q = 2$ and $p = q = 3$.

Theorem 4.1. For $\mu = (p, q, 1^{n-p-q})$ we have

$$\begin{aligned} \frac{n!}{(n-p-q)!} \hat{\chi}_\mu^\lambda &= \sum_{i,j \geq 2} c_i^\lambda(p) c_j^\lambda(q) s(p+1, i) s(q+1, j) \\ &+ pq \sum_{i,j \geq 0} c_i^\lambda(p) c_j^\lambda(q) \left(\sum_{a,b \geq 0} \binom{a+1}{b+1} \frac{p(-p)^b + q^{b+1}}{p+q} s(p+1, i-a+b) s(q+1, j+a+2) \right). \end{aligned}$$

Observe that the above expression is not symmetrical with respect to p, q , though actually $\hat{\chi}_\mu^\lambda$ is. For $q = 1$ the sums are restricted to $j = 2$ and $j = a = b = 0$, respectively, thus we recover Theorem 3.1. As an example, for $\mu = (4, 3, 1^{n-7})$ we have

$$\begin{aligned} \frac{n!}{(n-7)!} \hat{\chi}_\mu^\lambda &= (c_5^\lambda(4) - 10c_4^\lambda(4) + 35c_3^\lambda(4) - 50c_2^\lambda(4))(c_4^\lambda(3) - 6c_3^\lambda(3) + 23c_2(3)) \\ &- 12(c_7^\lambda(4) + 15c_6^\lambda(4) - 95c_5^\lambda(4) + 325c_4^\lambda(4) - 624c_3^\lambda(4) + 620c_2^\lambda(4)). \end{aligned}$$

5. The general case

Theorem 4.1. may be written in a more compact form, by using the following notations. Let $\varepsilon \in \{0, 2\}$. Define $\theta = 1$ if $\varepsilon = 0$ and $\theta = pq$ otherwise. Then Theorem 4.1. can be expressed

$$\begin{aligned} \frac{n!}{(n-p-q)!} \hat{\chi}_\mu^\lambda &= \sum_{\varepsilon \in \{0,2\}} \sum_{i,j \geq 0} c_i^\lambda(p) c_j^\lambda(q) \\ &\times \left(\sum_{a,b \geq 0} \theta \binom{a+1}{b+1} \frac{p(-p)^b + q^{b+1}}{p+q} s(p+1, i-a+b) s(q+1, j+a+\varepsilon) \right) \end{aligned}$$

with the convention that the sum on a, b is restricted to $a = b = 0$ when $\varepsilon = 0$.

A similar notation will be useful in the general case. For $p = (p_1, \dots, p_r) \in \mathbb{N}^r$ write $|p| = \sum_i p_i$. Let $M^{(r)}$ denote the set of upper triangular $r \times r$ matrices with nonnegative integers, and 0 on the diagonal. For any $1 \leq i < j \leq r$ let $\varepsilon_{ij} \in \{0, 2\}$, and define $\theta_{ij} = 1$ if $\varepsilon_{ij} = 0$ and $\theta_{ij} = p_i p_j$ otherwise.

Theorem 5.1. For $\mu = (p_1, \dots, p_r, 1^{n-|p|})$ we have

$$\begin{aligned} \frac{n!}{(n-|p|)!} \hat{\chi}_\mu^\lambda &= \sum_{\varepsilon \in \{0,2\}} \sum_{\substack{r(r-1) \\ 2}} \sum_{(i_1, \dots, i_r) \in \mathbb{N}^r} \sum_{a, b \in M^{(r)}} \prod_{k=1}^r c_{i_k}^\lambda(p_k) \\ &\times \left(\prod_{1 \leq i < j \leq r} \theta_{ij} \binom{a_{ij} + 1}{b_{ij} + 1} \frac{p_i(-p_i)^{b_{ij}} + p_j^{b_{ij}+1}}{p_i + p_j} \right) s \left(p_k + 1, i_k + \sum_{l < k} (a_{lk} + \varepsilon_{lk}) - \sum_{l > k} (a_{kl} - b_{kl}) \right), \end{aligned}$$

with the convention that the sum on a_{ij}, b_{ij} is restricted to $a_{ij} = b_{ij} = 0$ when $\varepsilon_{ij} = 0$.

For $p_r = 1$ either all ε_{lr} , $1 \leq l \leq r - 1$ are equal to 0, and the sum is restricted to $i_r = 2$. Either only one ε_{lr} is equal to 2, and the sum is restricted to $i_r = 0$. Or two or more ε_{lr} are equal to 2, such a contribution being zero. Summing up the r non zero contributions brings a factor $(n - \sum_{i=1}^{r-1} p_i)$, and we recover the formula for $r - 1$.

Ingram [4] had only computed the case $p_1 = p_2 = p_3 = 2$. As an example, for $\mu = (3, 2, 2, 1^{n-7})$ we have

$$\begin{aligned} \frac{n!}{(n-7)!} \hat{\chi}_\mu^\lambda &= c_2^\lambda(2)(36c_5^\lambda(3) - 338c_4^\lambda(3) + 1128c_3^\lambda(3) - 1558c_2^\lambda(3)) \\ &\quad - c_3^\lambda(2)(12c_5^\lambda(3) - 114c_4^\lambda(3) + 384c_3^\lambda(3) - 534c_2^\lambda(3)) \\ &\quad + (9c_2^\lambda(2)^2 - 6c_3^\lambda(2)c_2^\lambda(2) + c_3^\lambda(2)^2 - 4c_4^\lambda(2))(c_4^\lambda(3) - 6c_3^\lambda(3) + 11c_2^\lambda(3)) \\ &\quad + 72(c_6^\lambda(3) - 10c_5^\lambda(3) + 40c_4^\lambda(3) - 80c_3^\lambda(3) + 79c_2^\lambda(3)). \end{aligned}$$

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