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Differential Geometry

Stable tangential family germs and singularities of their envelopes

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Abstract

A tangential family is a system of regular curves emanating tangentially from another regular curve. We classify tangential family germs which are stable under deformations among tangential families and we study singularities of their envelopes. We also discuss some applications of our results to Differential Geometry. ***To cite this article: G. Capitanio, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***
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Résumé

Germes stables de familles tangentielles et singularités de leurs enveloppes. Une famille tangentielle est un système de courbes régulières, émanées tangentiellement par une autre courbe régulière. Nous classifions les germes de familles tangentielles qui sont stables par déformations parmi les familles tangentielles, et nous étudions les singularités des enveloppes correspondantes. Nous étudions aussi certaines applications de nos résultats en Géométrie Différentielle. ***Pour citer cet article : G. Capitanio, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***
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Version française abrégée

Une famille tangentielle est un système de courbes régulières émanées tangentiellement par une autre courbe régulière, appelée support. Les familles tangentielles apparaissent naturellement en Géométrie des Caustiques, Théorie des Singularités, Optique, Traitement des Images et Géométrie Différentielle. Par exemple, les géodésiques tangentes à une courbe dans une surface Riemannienne forment une famille tangentielle, de même que les courbes intégrales des équations différentielles de type Clairaut.

L'origine de ce sujet remonte à l'étude, mené par Huygens, des caustiques des rayons lumineux. La théorie moderne des enveloppes se base sur les travaux d'Arnold [1,2], Thom [9] et Dufour [6]. Notre approche est toutefois différente, ce qui nous permet d'étudier pour la première fois le cas d'enveloppes à plusieurs branches locales.

Les singularités simples des familles tangentielles (avec leurs déformations et les perestroikas des enveloppes correspondantes) ont été classifiées dans [4]. Les familles tangentielles ont été aussi étudiées du point de vue de la Géométrie de Contact dans [5]. Les familles tangentielles à support semicubique ont été étudiées dans [3].

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Considérons une application lisse $f:\mathbb{R}^2 \to \mathbb{R}^2$ du plan fibré $\mathbb{R}^2 = \mathbb{R}_1 \times \mathbb{R}_2 \to \mathbb{R}_1$, muni des coordonnées $\xi \in \mathbb{R}_1$ et $t \in \mathbb{R}_2$. L'application f définit une famille de courbes f_ξ, paramétrée par $\xi \in \mathbb{R}$, où $f_\xi(t) = f(\xi, t)$.

Définition 0.1. Une famille de courbes régulières $\{f_\xi\colon \xi \in \mathbb{R}\}$ est dite *tangentielle* si la courbe $f_{|t=0}$, appelée *support*, est régulière et chaque courbe f_ξ est tangente au support au point $f(\xi, 0)$.

Le *graphe* d'une famille tangentielle est la surface $\Phi := \{(\xi, f(\xi,t))\colon \xi, t \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}^2$. L'enveloppe de la famille est le contour apparent du graphe dans le plan (c'est-à-dire, l'ensemble des valeurs critiques de la restriction au graphe de la projection naturelle $\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$).

Pour l'étude locale, nous identifions les familles tangentielles aux germes (à l'origine) de leurs paramétrisations. Deux germes à l'origine de familles tangentielles f et g sont dits $\mathcal{A}$-*équivalents* si on peut transformer l'un dans l'autre par des changements de coordonnées des espaces source et but préservant les respectives origines (la structure fibré de l'espace source n'étant pas forcément préservée).

Une *déformation tangentielle* à p paramètres ($p \in \mathbb{N} \cup \{0\}$) d'une famille tangentielle f est une application $F:\mathbb{R}^2 \times \mathbb{R}^p \to \mathbb{R}^2$ telle que $F_{|\lambda=0} = f$ et $F(\cdot, \lambda)$ est une famille tangentielle pour tout λ. On a une définition analogue dans le cadre local.

Le germe (à l'origine) d'une famille tangentielle f est *stable* si pour toute déformation tangentielle F de f, $F_{|\lambda=\lambda_0}$ est $\mathcal{A}$-équivalent au germe de f à l'origine en un point, dépendant de λ_0, arbitrairement proche de l'origine pour λ_0 suffisamment petit.

Le but de cette Note est de classifier à $\mathcal{A}$-équivalence près les germes stables de familles tangentielles (par rapport aux petites déformations tangentielles). Notre résultat principal est le suivant.

Théorème 0.2. *Le germe d'une famille tangentielle est stable esi et seulement si il est $\mathcal{A}$-équivalent au germe à l'origine de la famille des droites tangentes à la parabole $y = x^2$ ou à la parabole cubique $y = x^3$.*

Comme les enveloppes de familles tangentielles $\mathcal{A}$-équivalentes sont difféomorphes, on déduit de ce théorème que l'auto-tangence d'ordre 2 est la seule singularité stable des enveloppes des familles tangentielles (par rapport aux déformations tangentielles). Dans le cadre global, il faut encore ajouter les autres singularités stables des enveloppes, à savoir, les auto-intersections transverses et les points de rebroussement ordinaires.

Ce résultat a plusieurs conséquences dans le cas des géodésiques tangentes à une courbe dans une surface Riemannienne (M, g).

Théorème 0.3. *Génériquement, l'enveloppe des géodésiques tangentes à une courbe régulière $\gamma : S^1 \to M$ est une courbe* (*contenant* $\gamma(S^1)$), *dont les singularités peuvent être des auto-intersections transverses, des points de rebroussement ordinaires et des auto-tangences d'ordre* 2. *Une telle enveloppe est stable sous petites déformations de la métrique g.*

Ici, « génériquement » signifie que les applications vérifiant le théorème forment un sous ensemble ouvert dense de $C^\infty(S^1, M)$ pour la topologie de Whitney.

D'autres applications de la théorie des familles tangentielles conduisent à des variations sur le thème du dernier théorème géométrique de Jacobi.

1. Introduction

A tangential family is a system of regular curves emanating tangentially from another regular curve, called support. Tangential families naturally arise in the Geometry of Caustics, Singularity Theory, Optics, Image Treatment and Differential Geometry. For example, the tangent geodesics of a curve in a Riemannian surface define a tangential family, as well as the integral curves of Clairaut-type differential equations.

The roots of the theme go back to Huygens' investigation of caustics of rays of light. Thom showed in [9] that the singularities of envelopes of generic 1-parameter families of smooth plane curves are semicubic cusps and transversal self-intersections. Normal forms of generic families of plane curves near regular points of their envelopes have been found by Arnold (see [1,2]; see also Dufour [6]). However, our study differs from the approaches of Thom and Arnold.

For example, for the first time the situation when the support curve is not the only local component of the envelope is studied.

The aim of this Note is to classify the singularities of tangential family germs which are stable under small deformations among tangential families, and to study their envelopes. We prove that there exists only one local stable singularity of envelopes of tangential family germs, the second order self-tangency. Applications to families of tangent geodesics of a regular curve on a Riemannian surface are given.

Simple singularities of tangential family germs (together with their tangential deformations and corresponding envelope perestroikas) are classified in [4]. Tangential families are discussed from the Contact Geometry viewpoint in [5]. Envelopes of tangential families with cusped support are studied in [3].

2. Stable tangential family germs

Unless otherwise specified, all the objects considered below are supposed to be of class C^∞; by plane curve we mean a smooth map $\mathbb{R} \to \mathbb{R}^2$. Let us consider a smooth map $f : \mathbb{R}^2 \to \mathbb{R}^2$ of the fibered plane $\mathbb{R}^2 = \mathbb{R}_1 \times \mathbb{R}_2 \to \mathbb{R}_1$, equipped with coordinates $\xi \in \mathbb{R}_1$ and $t \in \mathbb{R}_2$. We set $f_\xi(t) = f(\xi, t)$.

Definition 2.1. A family of regular curves $\{f_\xi\colon\ \xi \in \mathbb{R}\}$ is a *tangential family* if the curve $f_{|t=0}$, called *support*, is regular and for every $\xi \in \mathbb{R}$, the curve f_ξ is tangent to the support at $f(\xi, 0)$.

The *graph* of the tangential family is the immersed surface $\Phi := \{(\xi, f(\xi, t))\colon\ \xi, t \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}^2$. Let us consider the two natural projections of Φ on $\mathbb{R}$ and $\mathbb{R}^2$, $\pi_1 : (q, p) \mapsto q$ and $\pi_2 : (q, p) \mapsto p$. The first projection π_1 is a fibration; the images by π_2 of its fibers are the curves of the family. The *envelope* of the family is the apparent contour of its graph in the plane (i.e., the critical value set of the restriction of π_2 to the graph). By the very definition, the support of a tangential family belongs to its envelope.

Our study of tangential families being local, we identify them to smooth map germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. Note that the graphs of tangential family germs are germs of embedded surfaces.

Let $\mathfrak{m}_{s,t}$ be the space of function germs $(\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ (when the structure of fiber bundle on the source plane is not taken into account we shall use s and t as coordinates, leaving ξ for the fibered case). The group $\mathcal{A} := \mathrm{Diff}(\mathbb{R}^2, 0)^2$ acts on $(\mathfrak{m}_{s,t})^2$ by the rule $(\phi, \psi) \cdot f := \psi \circ f \circ \phi^{-1}$. Two map germs of $(\mathfrak{m}_{s,t})^2$ are $\mathcal{A}$-*equivalent* if they belong to the same $\mathcal{A}$-orbit.

Since $\mathcal{A}$-equivalence does not preserve the fibered structure of $\mathbb{R}^2 = \mathbb{R}_1 \times \mathbb{R}_2$, $\mathcal{A}$-equivalent map germs may parameterize non-diffeomorphic families of curves. However, this equivalence preserves the envelopes of these families. Indeed, we obtain from the chain rule and Cauchy–Binet Theorem the following result.

Proposition 2.2. *Critical value sets of $\mathcal{A}$-equivalent map germs are diffeomorphic*; *in particular, envelopes of $\mathcal{A}$-equivalent tangential families are diffeomorphic.*

Remark 1. The more natural equivalence relation (as in [6]), preserving the fibered structure of the mappings' source plane, is too much fine: the fold is the only stable and simple singularity, all the other singularities having functional moduli.

In some situations, as for instance in the study of geodesic tangential family evolution under small perturbations of the metric, it would be natural to perturb a tangential family only among tangential families.

Definition 2.3. Let $p \in \mathbb{N} \cup \{0\}$. A p-parameter *tangential deformation* of a tangential family f is a deformation $F : \mathbb{R}^2 \times \mathbb{R}^p \to \mathbb{R}^2$ of f (i.e. $F_{|\lambda=0} = f$) such that $F(\cdot\,; \lambda)$ is a tangential family for every λ.

An analogous definition holds for germs. Note that a tangential deformation induces a smooth deformation on the family support.

A tangential family germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is said to be *stable* if for every tangential deformation F of f, $F_{|\lambda=\lambda_0}$ is $\mathcal{A}$-equivalent to f at some λ_0-depending point, arbitrary close to the origin for λ_0 small enough.

We state now our main result, whose proof is given in Section 4.

Theorem 2.4. *A tangential family germ is stable if and only if it is $\mathcal{A}$-equivalent to the germ at the origin of the family of the tangent lines of the parabola $y = x^2$ or of the cubic parabola $y = x^3$.*

The first normal form is the fold singularity; the corresponding envelope is smooth.

Corollary 2.5. *The second order self-tangency is the only stable singularity of envelopes of tangential family germs* (*under small tangential deformations of the families*).

Remark 2. This singularity is not stable under non-tangential deformations. Under such a deformation the family's envelope experiences a beak to beak perestroika. From the global viewpoint, one has to add standard stable singularities of envelopes, namely semicubic cusps and transversal self-intersections.

3. Examples and applications

Every regular curve in a Riemannian surface (M, g) defines the *geodesic tangential family* formed by its tangent geodesics. We call *geodesic envelope of* γ the envelope of the geodesic tangential family of support γ.

For simplicity, we assume M closed. Generically, the only singularities of a regular curve $\gamma : S^1 \to M$ are transversal self-intersections. Here (and below) 'generically' means that the mappings satisfying the claim form an open dense subset of $C^\infty(S^1, M)$ for the Whitney topology.

A deformation of the metric g induces a tangential deformation on the family (fixing the support), as well as a deformation of γ.

Theorem 3.1. *The geodesic envelope of a regular curve $\gamma : S^1 \to M$ is generically a curve* (γ *being one of its components*)*, whose singularities may be only transversal self-intersections, semicubic cusps and order* 2 *self-tangencies. Such an envelope is stable under small enough perturbations of γ and of the metric g.*

The theorem follows from Theorem 2.4 and Thom's Transversality Lemma. A similar statement holds if we replace the Riemannian structure with a projective structure.

We discuss now an example of a global geodesic tangential family whose envelope has infinitely many branches. Let us consider in the standard Euclidean space $\mathbb{R}^3$ the sphere S^2, equipped with the induced metric. Let $\gamma_r \subset S^2$ be a circle of radius r in $\mathbb{R}^3$ and let $f : S^1 \times \mathbb{R} \to S^2$ be a parameterization of the geodesic tangential family of support γ_r in S^2, where for every $\xi \in S^1 \simeq \gamma_r$, the map $f_\xi : \mathbb{R} \to S^2$ is a 2π-periodical parameterization of the S^2 great circle tangent to γ_r at ξ.

The family envelope has two branches, γ_r and its opposite circle, each one having infinite multiplicity. Indeed, the critical set of the parameterization, given by the equation $\det Df(\xi, t) = 0$, has infinitely many disjoint components $C_n := (\xi \in S^1, t = n\pi)$ in the cylinder $S^1 \times \mathbb{R}$, where $n \in \mathbb{Z}$. The images $E_n := f(C_n)$ of these components are called the *order n envelopes* of γ_r. Notice that E_n is γ_r if n is even, the opposite circle if n is odd.

Let $\widetilde{S}^2$ be a small perturbation of the sphere. Denote by $\tilde{\gamma}_r$ the support of the perturbed family and by $\tilde{f} : S^1 \times \mathbb{R} \to \widetilde{S}^2$ the perturbed geodesic tangential family.

Theorem 3.2. *Fix $n_0, m_0 \in \mathbb{N}$ arbitrary large, $n_0 < m_0$. If $\widetilde{S}^2$ is close enough to the initial sphere S^2, then the order n envelopes of $\tilde{\gamma}_r$ are generic spherical closed caustics with zero Maslov number for $|n| < m_0$, smooth for $|n| < n_0$.*

If $\widetilde{S}^2$ is close enough to S^2, for every $n < m_0$ the components of the $\tilde{f}$-critical set, denoted by $\widetilde{C}_n$, are pairwise disjoint small perturbations of the curves C_n. Each order n envelope is a small perturbation of the corresponding order n unperturbed envelope. Since the unperturbed tangential family is equivalent to a fold at every point of its envelope, by Theorem 2.4 the perturbed family is $\mathcal{A}$-equivalent to a fold at every point of the envelope branches $\widetilde{E}_n$ for every $|n| < n_0$, provided that $\widetilde{S}^2$ is close enough to S^2.

The perturbed envelope $\widetilde{E}_n$ can be viewed as a caustic, images of $\widetilde{E}_{n-1}$ under a geodesic flow. Hence, generic singularities of caustics, as semicubic cusps and transversal self-intersection, may arise in envelopes $\widetilde{E}_n$ for $n_0 < n < m_0$. The theorem is proven.

Consider now a fixed arbitrary small perturbation $\widetilde{S}^2$ of S^2.

Theorem 3.3. *The first order envelopes of $\tilde{\gamma}_r$, $E_{\pm 1}(\gamma)$, have each one at least four cusps, provided that $\widetilde{S}^2$ is a generic surface close enough to the sphere and r is small enough.*

First deform slightly the sphere to an ellipsoid, arbitrary close both to S^2 and $\widetilde{S}^2$. When $r \to 0$, the corresponding perturbed curve on the ellipsoid $\tilde{\gamma}_r$ shrinks about a point and the first order envelope branches approach the first caustics of this point. Then the Last Geometrical Theorem of Jacobi states that the first caustic of a point has at least 4 cusps (see [7]).

Now deform the ellipsoid to the generic surface $\widetilde{S}^2$, arbitrary close to it. The theorem follows now from the envelope stability under small enough tangential deformations.

4. Proof of Theorem 2.4

In order to prove Theorem 2.4, we start computing a standard (not unique) 'prenormal form' to which is possible to bring every tangential family germ by $\mathcal{A}$-equivalence.

Lemma 4.1. *The 3-jet of every tangential family germ is $\mathcal{A}$-equivalent to $(s,t) \mapsto (s, at^2 + bt^3 + cst^2)$, for some $a, b, c \in \mathbb{R}$.*

Consider a tangential family germ at the origin, and fix a local coordinate system $\{x, y\}$ in which the family support has equation $y = 0$. Denote by f_ξ the family curve corresponding to the support point $(\xi, 0)$. Define k_0 and k_1 by the expansion $k_0 + k_1\xi + o(\xi)$ (for $\xi \to 0$) of half of the curvature of f_ξ at $(\xi, 0)$; α is similarly defined by the expansion $k_0t^2 + \alpha t^3 + o(t^3)$ for $t \to 0$ of the function, whose graph is the image of f_0. For any small enough value of ξ, f_ξ can be parameterized near $(\xi, 0)$ by its projection $\xi + t$ on the x-axis. This way we get a family's parameterization, whose 3-jet is $(t + \xi, k_0t^2 + k_1t^2\xi + \alpha t^3)$. Taking $\xi + t$ and t as new parameters, we obtain the required germ, with $a = k_0$, $b = \alpha - k_1$ and $c = k_1$. The lemma is proven.

We can start now the proof of Theorems 2.4. Consider a tangential family germ, whose 3-jet is $\mathcal{A}$-equivalent to $(s, at^2 + bt^3 + cst^2)$. If $a \neq 0$, its 2-jet is $\mathcal{A}$-equivalent to a fold (s, t^2), which is 2-$\mathcal{A}$-determined and $\mathcal{A}$-stable [8]. Notice that the fold map germ is $\mathcal{A}$-equivalent to the germ at the origin of the family of the tangent lines of the parabola $y = x^2$, which is parameterized by $(\xi, t) \mapsto (\xi + t, \xi^2 + 2\xi t)$.

We suppose from now on $a = 0$ and $bc \neq 0$. By rescaling we may assume $b = c = 1$. In this case, it is well known that this 3-jet is $\mathcal{A}$-sufficient (see e.g. [8]). Its $\mathcal{A}$-orbit contains Theorem 2.4 normal form $h(\xi, t) := (\xi + t, \xi^3 + 3\xi t^2)$, parameterizing the family of the tangent lines to $y = x^3$. On the other hand, a miniversal tangential deformation of h is provided by $F(\xi, t; \lambda) = h(\xi, t) + \lambda(0, t)$.

Let us consider now a tangential family germ f, $\mathcal{A}$-equivalent to h. By the very definition of versality, any p-parameter tangential deformation of f can be represented as

$$(\mathbb{R}^2 \times \mathbb{R}^p, 0) \ni (\xi, t; \mu) \mapsto \Psi\big(F\big(\Phi(\xi, t; \mu), \tilde{\lambda}(\mu)\big), \mu\big) \in (\mathbb{R}^2, 0),$$

where $\Phi(\cdot\,; \mu)$ and $\Psi(\cdot\,; \mu)$ are deformations of the identity diffeomorphisms of the source and the target planes $\mathbb{R}^2$ and $\tilde{\lambda}$ is a function germ $(\mathbb{R}^p, 0) \to (\mathbb{R}, 0)$.

Assume $\tilde{\lambda} \not\equiv 0$; then there exists an arbitrary small μ_0 for which $\tilde{\lambda}(\mu_0) \neq 0$. Hence, when μ goes from 0 to μ_0, the envelope of the deformed family experiences a beaks perestroika. The considered deformation being tangential, this is impossible. Thus, $\tilde{\lambda} \equiv 0$ and the deformation is trivial. This proves that every tangential family germ $\mathcal{A}$-equivalent to h is stable.

Finally, by the above lemma, the 3-jet of every tangential family germ, $\mathcal{A}$-equivalent neither to a fold nor to h, is $\mathcal{A}$-equivalent to $(\xi + t, b(\xi + t)t^2 + ct^3)$ with $bc = 0$. The deformation $f(\xi, t) + \lambda(0, \xi t^2)$ is clearly tangential and non-trivial. Indeed, the perturbed prenormal form's coefficients are $b(\lambda) = \alpha - k_1 - \lambda$ and $c(\lambda) = k_1 + \lambda$. Their product is nonvanishing for every $\lambda \neq 0$ small enough. This shows that such a germ is not stable and completes the proof.

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