Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
C. R. Acad. Sci. Paris, Ser. I 341 (2005) 475–480
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Mathematical Analysis

Reiter's condition (P_2) and hypergroup representations

Liliana Pavel

Faculty of Mathematics and Computer Science, University of Bucharest, Academiei 14, 010014 Bucharest, Romania

Received 25 July 2005; accepted 1 September 2005

Presented by Jean-Pierre Kahane

Abstract

Let K be a hypergroup with Haar measure. We relate Reiter's condition (P_2) to positive definite measures and associated representations, induced representations from a subgroup of K and convolution operators on $L_2(K)$. ***To cite this article: L. Pavel, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***
© 2005 Académie des sciences. Published by Elsevier SAS. All rights reserved.

Résumé

La condition (P_2) de Reiter et les représentations des hypergroupes. Soit K un hypergroupe qui possède une mesure invariante à gauche. Nous établissons la connexion entre la condition (P_2) de Reiter et les représentations associées aux mesures de type positif, représentations induites d'un sous-groupe de K et d'opérateurs de convolution sur $L_2(K)$. ***Pour citer cet article : L. Pavel, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***
© 2005 Académie des sciences. Published by Elsevier SAS. All rights reserved.

Version française abrégée

Les hypergroupes généralisent les groupes localement compacts. Plusieurs notions et résultats de l'analyse harmonique et de la théorie des représentations des groupes localement compacts peuvent être transposés aux hypergroupes sans difficulté, mais il y a des sujets où les différences sont notables. La condition (P_2) a été introduite par Reiter [13] dans le cas des groupes et généralisée aux hypergroupes par Skantharajah [15] : un hypergroupe K satisfait à la condition (P_2) de Reiter si pour tout ensemble compact $E \subseteq K$ et $\varepsilon > 0$ il existe une fonction $\varphi \in L_2(K)$, $\varphi \geqslant 0$, $\|\varphi\|_2 = 1$ telle que $\|\delta_x * \varphi - \varphi\|_2 < \varepsilon$, $\forall x \in E$. Skantharajah a démontré que les hypergroupes qui satisfont à cette condition sont moyennables, mais contrarement au cas des groupes localement compacts, la réciproque n'est pas valide. Un rôle important dans notre note est joué par la notion de représentation subordonnée à l'autre (introduite de Fell [4] dans le cas des groupes et translatée littéralement aux hypergroupes) : la représentation π est faiblement contenue dans la représentation τ (noté $\pi \preccurlyeq \tau$) si toutes les fonctions de type positif associées à π sont limites pour la convergence compacte sur K de sommes de fonctions de type positif associées à τ. Nous considèrons aussi la représentation constante de dimension 1, i_K et la représentation regulière à gauche, λ_K de K sur $L_2(K)$: $i_K(\mu)(f) = f$ et $\lambda_K(\mu)(f) = \mu * f$, $\forall f \in L_2(K)$, $\forall \mu$ mesure bornée. Le dual de K, l'ensemble des représentations irréducibles

E-mail address: lpavel@fmi.unibuc.ro (L. Pavel).

1631-073X/$ – see front matter © 2005 Académie des sciences. Published by Elsevier SAS. All rights reserved.
doi:10.1016/j.crma.2005.09.009

de K sera noté par $K^\wedge$ et le dual reduit de K, l'ensemble des représentations irréducibles de K subordonnées à λ_K, par $K_r^\wedge$.

Le but de cette Note est de contribuer à l'étude des représentations des hypergroupes en établissant la connexion entre la condition (P_2) de Reiter et certaines directions de la théorie des représentations qui sont reliées habituellement, dans le cas des groupes localement compacts, à la classe des groupes moyennables (voir [12]).

Premièrement nous étudions $K_r^\wedge$ (Théorème 3.1) et démontrons qu'un hypergroupe satisfait à la condition (P_2) de Reiter si et seulement si $i_K \preccurlyeq \lambda_K$. Les résultats principaux Théorème 3.4, Théorème 3.5 et Théorème 3.6 sont des extensions aux hypergroupes des théorèmes concernant les groupes localement compacts moyennables obtenus par Skudlarek [16], Greanleaf [8] et Granirer [7] respectivement. Dans nos démonstrations la condition (P_2) de Reiter joue le rôle dévolu à la moyennabilité. Nous avons prouvé :

Théorème 0.1. *Soit K un hypergroupe commutatif qui satisfait à la condition (P_2) de Reiter et μ une mesure de type positif bornée sur K telle que $\mu(K) \neq 0$. Alors $\iota_K \preccurlyeq T^\mu$.*

Dans ce théorème T^μ est la représentation associées à μ (voir version anglaise).

Théorème 0.2. *Soit K un hypergroupe. Alors K satisfait à la condition (P_2) de Reiter si et seulement si pour toute représentation $\pi \in K^\wedge$ et pour tout sous-groupe fermé H de K on a $\pi \preceq \mathrm{ind}_{H \nearrow K}\, \pi|_H$.*

Pour la construction de la représentation induite d'un sous-groupe à l'hypergroupe on peut consulter [9].

Théorème 0.3. *Soit K un hypergroupe qui satisfait à la condition (P_2) de Reiter. Alors, l'adhérence faible de l'ensemble $\{\lambda_K(\mu) \mid \mu$ mesure de probabilité$\}$ dans l'espace des opérateurs bornés sur $L_2(K)$ ne contient pas l'opérateur nul.*

1. Introduction

Hypergroups are locally compact spaces whose bounded Radon measures form an algebra which has properties similar to the convolution measure algebra of a locally compact group. A hypergroup can be viewed as a probabilistic group in the sense that to each pair $x, y \in K$ there exists a probability measure $\delta_x * \delta_y$ on K with compact support, such that $(x, y) \mapsto \operatorname{supp} \delta_x * \delta_y$ is a continuous mapping from $K \times K$ into the space of compact subsets of K. Unlike the groups, $\delta_x * \delta_y$ is not in general a point measure. The convolution between point measures extends to all bounded Radon measures on K. Important examples of hypergroups are double coset spaces, orbit spaces and duals arising from certain classes of locally compact groups and corresponding topological group actions. The substantial development of the theory of hypergroups with the works of Dunkl [3], Spector [17] and Jewett [11] put hypergroups in the right setting for harmonic analysis and representations theory. As hypergroups generalize locally compact groups, many basic notions and facts from harmonic analysis and representations theory of groups carry over to hypergroups only with minor changes, but there are certain topics in which the differences are substantial.

The condition (P_2) was introduced by Reiter [13] in the groups case and later translated to hypergroups by Skantharajah [15]: a hypergroup K satisfies the condition (P_2) if whenever $\varepsilon > 0$ and a compact set $E \subseteq K$ are given, there exists a function $\varphi \in L_2(K)$, $\varphi \geqslant 0$, $\|\varphi\|_2 = 1$ such that $\|\delta_x * \varphi - \varphi\|_2 < \varepsilon$ for every $x \in E$. Skantharajah [15, Lemma 4.4] proved that, similar to the groups case, this condition means that 1_K is the uniform limit on every compact subset in K of functions of the form $\varphi * \varphi^\sim$, where $\varphi \in L_2(K)$, $\varphi \geqslant 0$, $\|\varphi\|_2 = 1$. He also proved that all hypergroups satisfying this condition are amenable, but different from the groups case there exist amenable hypergroups which do not satisfy it (see [15, Example 4.6]). A hypergroup K is said to be amenable if there exists a left invariant mean on $L_\infty(K)$.

The purpose of this Note is to relate Reiter's condition (P_2) on hypergroups to those directions of the representations theory which in the context of locally compact groups are usually linked to amenability (see [12]): positive definite measures and associated representations, induced representations and convolution operators. After reformulating Reiter's condition (P_2) in representations terms, we prove the main theorems: a weak containment result for representations associated to positive definite measures, a Frobenius reciprocity theorem for induced representations from a subgroup of the hypergroup and a characteristic property of the weak closure of the image of the left regular representation. We partially extend from locally compact groups to hypergroups results of Skudlarek [16], Greenleaf [8] and Granirer [7], respectively. In our approach the role of amenability is played by Reiter's condition (P_2).

2. Preliminaries

For basic definitions and results on hypergroups we shall follow [11]. K always stands for a hypergroup with a fixed Haar measure m, symbols like $\int \cdots \mathrm{d}x$ will always denote integration with respect to m. $\mathcal{M}(K)$ is the space of all bounded regular (complex valued) Borel measures on K. Our notation generally agrees with [11]. However the following notations are different from [11]: δ_x denotes the Dirac measure concentrated at x and $x \mapsto x^\vee$ the involution on K. We mention that $e \in \operatorname{supp} \delta_x * \delta_y$ if and only if $x = y^\vee$ and that the involution naturally extends to all positive measures on K by $(\delta_x)^\vee = \delta_{x^\vee}$. If f is a Borel function on K and $x, y \in K$ the left translate f_x is defined on K by $f_x(y) = \int f \,\mathrm{d}\delta_x * \delta_y = f(x * y)$, if the integral exists. The function $f^\sim$ on K is given by $f^\sim(x) = \overline{f(x^\vee)}$. Convolution of two functions f and g on K is defined by $(f * g)(x) = \int f(x * y) g(y^\vee)\,\mathrm{d}y$, whenever it makes sense. If $\mu \in \mathcal{M}(K)$ and f is a Borel function, then the convolution $\mu * f$ is defined on K by $(\mu * f)(x) = \int f(y^\vee * x)\,\mathrm{d}\mu(y)$.

Following [11], we define a representation of K as a norm increasing $*$-representation of the Banach $*$-algebra $\mathcal{M}(K)$. For notational convenience, we write π_x for $\pi(\delta_x)$, where $x \in K$. Just as for the locally compact groups case, to any representation π of K and any vector ξ in $\mathcal{H}^\pi$, corresponds a continuous positive definite function, necessarily bounded, $f_\xi^\pi(x) = \langle \pi_x \xi, \xi \rangle$. The one trivial representation and the left regular representation of K on $L_2(K)$ will be denoted by i_K and λ_K, respectively; so $i_K(\mu)(f) = f$ and $\lambda_K(\mu)(f) = \mu * f$ for each $f \in L_2(K)$ and $\mu \in \mathcal{M}(K)$.

Similar to the locally compact groups case, $L_1(K)$ is an involutive Banach algebra admitting approximate units. We denote by $C^*(K)$ its enveloping algebra (see [2, 2.7]). As the left regular representation of K is faithful, we can embed $L_1(K)$ (as well as $\mathcal{C}_c(K)$) in $C^*(K)$. The (irreducible) representations of K, are in one to one correspondence with the non-degenerate (irreducible) representations of $L_1(K)$ and hence of $C^*(K)$. Thus, we are enabled to identify the dual $K^\wedge$ with $C^*(K)^\wedge$, each of the sets being endowed with the hull kernel topology [4]. Corresponding representations of the objects K, $C^*(K)$, will be denoted by the same letter. For $\pi, \sigma \in K^\wedge$, we say that π is weakly contained in σ if all positive functionals associated with π can be weakly*-approximated by sums of positive functionals associated with σ; in this case we write $\pi \preccurlyeq \sigma$. Recall that $\operatorname{supp} \sigma = \{\pi \in K^\wedge \mid \pi \preccurlyeq \sigma\}$. We will denote the support of λ_K by $K_r^\wedge$ and we will call it the reduced dual of K. In the commutative case $K_r^\wedge$ is actually the support of the Plancherel measure on $K^\wedge$, object of thorough studies in recent works by Lasser, Filbir and Voit (see for example [6], [18]).

3. Main results

Extending the well known result that characterizes the elements of the reduced dual of a locally compact group, we have:

Theorem 3.1. *Let K be a hypergroup and $\pi \in K^\wedge$. Then $\pi \in K_r^\wedge$ if and only if for each unit vector $\xi \in (\mathcal{H}^\pi, \langle\cdot,\cdot\rangle)$ there exists a net $(\varphi_i)_{i \in I} \subseteq L_2(K)$, $\varphi_i \geqslant 0$, $\|\varphi_i\|_2 = 1$ such that $\varphi_i * \varphi_i^\sim$ converges to the function f_ξ^π uniformly on compact subsets of K.*

Proof. We notice that the coefficients of the left regular representation of K are given by

$$\big\langle (\lambda_K)_x \varphi, \varphi \big\rangle = \int_K \varphi_{x^\vee}(y) \overline{\varphi(y)}\,\mathrm{d}y = (\varphi * \varphi^\sim)(x), \quad x \in K,$$

with $\varphi \in L_2(K)$. It is known (see for example [9]) that on the set of all normalized continuous bounded positive definite functions on K the weak*-topology coincides with the topology of uniform convergence on compact sets. Then, clearly, $\pi \preccurlyeq \lambda_K$ if and only if any positive definite function associated to π is the uniform limit on the compact subsets of K of positive definite functions arising from the left regular representation. □

Corollary 3.2. *A hypergroup K satisfies Reiter's condition (P_2) if and only if $i_K \in K_r^\wedge$.*

Proof. It follows immediately from Theorem 3.1 and [15, Lemma 4.4]. □

Remark 1. In their paper [6], Filbir and Lasser have introduced in the context of commutative hypergroups a concept useful for investigating the support of the Plancherel measure. They defined for each $\alpha \in K^\wedge$ the so-called "(P_2) condition satisfied in α" and proved that it is equivalent to α is in the support of the Plancherel measure. "The (P_2) condition is satisfied in 1_K" means that Reiter's condition (P_2) holds. In order to find sufficient conditions for $\pi \in K_r^\wedge$ by analogy to the commutative case [6], one can derive some kind of "condition (P_2) satisfied in $\pi \in K^\wedge$", as follows: we say that the (P_2) condition is satisfied in π if for each unit vector $\xi \in (\mathcal{H}^\pi, \langle\cdot,\cdot\rangle)$, whenever $\varepsilon > 0$ and a compact set $E \subseteq K$ are given, there exists a function $\varphi \in L_2(K)$, $\|\varphi\|_2 = 1$ such that $\|\delta_x * \varphi - f_\xi^\pi(x^\vee)\varphi\|_2 < \varepsilon$, $\forall x \in E$.

It is immediate that if the (P_2) condition is satisfied in π, then $\pi \in K_r^\wedge$. Indeed, let $\varepsilon > 0$ and E a compact subset of K. Then, there exists $\varphi \in L_2(K)$, $\varphi \geqslant 0$, $\|\varphi\|_2 = 1$ such that $\|\delta_x * \varphi - f_\xi^\pi(x^\vee)\varphi\|_2 < \varepsilon$, $\forall x \in E$. Hence, with the Cauchy–Schwarz inequality, we have:

$$\begin{aligned}\left|f_\xi^\pi(x^\vee) - \varphi * \varphi^\sim(x)\right| &= \left|f_\xi^\pi(x^\vee)\int_K \varphi(y)\varphi(y)\,\mathrm{d}y - \int_K \varphi(y)\varphi(x^\vee * y)\,\mathrm{d}y\right| \\ &= \left|\int_K \varphi(y)\left[f_\xi^\pi(x^\vee)\varphi(y) - \varphi(x^\vee * y)\right]\mathrm{d}y\right| = \left|\left\langle f_\xi^\pi(x^\vee)\varphi - \varphi_{x^\vee}, \varphi\right\rangle_2\right| \\ &\leqslant \|\varphi\|_2 \left\|\delta_x * \varphi - f_\xi^\pi(x^\vee)\varphi\right\|_2, \quad \forall x \in E.\end{aligned}$$

The next theorems show that Reiter's condition (P_2) is deeply involved in the representation theory of hypergroups, playing the role of the amenability in the locally compact groups context.

We recall that a measure $\mu \in \mathcal{M}(K)$ is positive definite on K if $\int_K f * f^\sim \,\mathrm{d}\mu \geqslant 0$, $\forall f \in \mathcal{C}_c(K)$.

Proposition 3.3. *Let K be a hypergroup, $\mu \in \mathcal{M}(K)$ a positive definite measure on K, and $\pi \in K_r^\wedge$. Then, for each positive definite function arising from π, f_ξ^π, $\int_K f_\xi^\pi \,\mathrm{d}\mu \geqslant 0$. In particular if K is a hypergroup satisfying the Reiter condition (P_2), then $\mu(K) \geqslant 0$.*

Proof. Since $\pi \preccurlyeq \lambda_K$, using Theorem 3.1 we have that f_ξ^π is a uniform limit on every compact subset in K of functions of the form $\varphi * \varphi^\sim$, $\varphi \in \mathcal{C}_c(K)$. As μ is positive definite, everything is clear. □

In [14], it is noticed that if the positive definite measure μ satisfies the so called condition (B), $\int_K f^\sim * g^\sim * g * f \,\mathrm{d}\mu \leqslant \|g\|_1^2 \int_K f * f^\sim \,\mathrm{d}\mu$, $\forall f, g \in \mathcal{C}_c(K)$, the usual Gelfand–Segal construction for the representation associated to μ can be applied. If K is commutative any positive definite measure $\mu \in \mathcal{M}(K)$ satisfies condition (B). Consequently, we can construct the associated representation of $L_1(K)$: for every $g \in \mathcal{C}_c(K)$, T_g^μ is the extension of $T_g^\mu([f]) = [g * f]$, $[f] \in \mathcal{C}_c(K)/\mathcal{N}^\mu(K)$, where $\mathcal{N}^\mu(K) = \{f \in \mathcal{C}_c(K) \mid \int_K f * f^\sim \,\mathrm{d}\mu = 0\}$, to the completion of $\mathcal{C}_c(K)/\mathcal{N}^\mu(K)$ with respect to the norm generated by the inner product $\langle [f], [g]\rangle = \int_K f * g^\sim \,\mathrm{d}\mu$. The next theorem translates to commutative hypergroups satisfying the (P_2) condition a well known property of amenable locally compact groups proven by Skudlarek [16].

Theorem 3.4. *Let K be a commutative hypergroup satisfying Reiter's condition (P_2) and $\mu \in \mathcal{M}(K)$ a positive definite measure on K with $\mu(K) \neq 0$. Then, $\iota_K \preccurlyeq T^\mu$.*

Proof. By Proposition 1, $\mu(K) > 0$. Let E be a compact subset of K and $\varepsilon > 0$. There exists a compact subset C in K such that $E \subset C$ and $|\mu|(K \setminus C) < \varepsilon/3$. By [11, 3.2], $C * C = \bigcup_{x,y \in C} \operatorname{supp} \delta_x * \delta_y$ is compact. We may suppose that C is symmetric, so since K satisfies the condition (P_2), we infer that there exists $\varphi \in \mathcal{C}_c(K)$, $\varphi \geqslant 0$, $\|\varphi\|_2 = 1$ such that $\|\varphi_{x^\vee * z} - \varphi\|_2 < \varepsilon\ (3\mu(K))^{-1}$, $\forall x, z \in C$. Then, using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\left|\left\langle T_x^\mu[\varphi], [\varphi]\right\rangle - \mu(K)\right| &= \left|\left\langle \int_K\int_K (\varphi_{x^\vee})_z(y)(\varphi^\sim)^\vee(y^\vee)\,\mathrm{d}y\,\mathrm{d}\mu(z)\right\rangle - \mu(K)\right| \\ &= \left|\int_{K\setminus C}\int_K \varphi\big(x^\vee * (z * y)\big)\varphi(y)\,\mathrm{d}y\,\mathrm{d}\mu(z) + \int_C\int_K \varphi\big(x^\vee * (z * y)\big)\varphi(y)\,\mathrm{d}y\,\mathrm{d}\mu(z)\right.\end{aligned}$$

$$- \int_C \int_K \varphi(y)\varphi(y)\,\mathrm{d}y\,\mathrm{d}\mu(z) + \int_C \int_K \varphi(y)\varphi(y)\,\mathrm{d}y\,\mathrm{d}\mu(z) - \mu(K)\Bigg|$$

$$\leqslant \Bigg|\int_{K\setminus C} \int_K \varphi\big(x^\vee * (z*y)\big)\varphi(y)\,\mathrm{d}y\,\mathrm{d}\mu(z)\Bigg| + \Bigg|\int_C \int_K \varphi\big(x^\vee * (z*y)\big)\varphi(y)\,\mathrm{d}y\,\mathrm{d}\mu(z)$$

$$- \int_C \int_K \varphi(y)\varphi(y)\,\mathrm{d}y\,\mathrm{d}\mu(z)\Bigg| + \Bigg|\int_C \int_K \varphi(y)\varphi(y)\,\mathrm{d}y\,\mathrm{d}\mu(z) - \mu(K)\Bigg|$$

$$\leqslant \|\varphi\|_2^2|\mu|(K\setminus C) + \|\varphi\|_2\varepsilon/3\mu(K)^{-1}\mu(K) + |\mu|(K\setminus C) < \varepsilon.$$

Hence, since $\mu(K)\cdot 1_K$ is approximated uniformly on the compact subsets by positive coefficients of the representation T^μ, it follows that $\iota_K \preccurlyeq T^\mu$. □

In [8], answering to a question raised by Fell [5], Greanleaf characterized the amenability of a locally compact group G by a certain Frobenius property: for each $\pi \in G^\wedge$ and every closed subgroup H of G, $\pi \preccurlyeq \mathrm{ind}_{H\nearrow G}\,\pi|_H$. This weak Frobenius reciprocity property can not be stated literally in the hypergroups context, replacing the subgroup H by a subhypergroup, because it is well known that not every representation of a subhypergroup H of a hypergroup K is inducible to K [10]. However, if H is a subgroup of the hypergroup K, then each representation of H is inducible to K. Consequently, focusing on this situation, we obtain:

Theorem 3.5. *Let K be a hypergroup. Then K satisfies Reiter's condition (P_2) if and only if for every $\pi \in K^\wedge$ and every closed subgroup H of K we have $\pi \preceq \mathrm{ind}_{H\nearrow K}\,\pi|_H$.*

Proof. If K satisfies the Reiter condition (P_2), the proof follows exactly as in [8, Theorem 5.1] for locally compact groups because the principal role in Greanleaf's approach is played by the Makey machinery for the construction of induced representations (which can be used in our situation, see [9]) and the amenability of the group in terms of Reiter's condition (P_2) (our hypothesis). The converse follows immediately, taking $H = \{e\}$ and $\pi = \iota_K$ because $\mathrm{ind}_{\{e\}\nearrow K}\,1_{\{e\}} = \lambda_K$. □

In the following theorem the closure of a subset Ω of the space of all bounded linear operators on $L_2(K)$ with respect to the weak operatorial topology will be denoted by ω-cl Ω.

Theorem 3.6. *Let K be a hypergroup which satisfies Reiter's condition (P_2). Then $0 \notin \omega\text{-cl}\{\lambda_K(\mu) \mid \mu$ probability measure$\}$.*

Proof. Let us denote by Ω the set $\{\lambda_K(\mu) \mid \mu$ probability measure$\}$. If $0 \in \omega$- cl Ω, then 0 is still in the norm closed convex hull of Ω. On the other hand, each T belonging to co Ω, $T = \sum_{j=1}^n c_j\lambda_K(\mu_j) = \lambda_K(\sum_{j=1}^n c_j\mu_j)$ has the norm equal to 1 as it follows from $\|\lambda_K(\mu)\| = \|\mu\|$, [15, Lemma 4.4] and from $\|\sum_{j=1}^n c_j\mu_j\| = 1$. □

Remark 2. Theorem 3.6 is motivated by a similar result from locally compact groups case obtained originally by Granirer [7] and used recently to give a characterization of the class of amenable locally compact groups [1, Theorem 2.7]. Hence, we might hope that it leads to a characterization result for hypergroups satisfying Reiter's condition (P_2).

Acknowledgements

The author was partially supported by the Romanian Academy under the grant No. 16-2005.

References

[1] C. Ching, X. Guangwu, The weak closure o the set of left translations operators, Proc. Amer. Math. Soc. 127 (1999) 465–471.
[2] J. Dixmier, Les C^*-algèbres et leurs représentations, Gauthier-Villars, Paris, 1964.

[3] C.F. Dunkl, The measure algebra of a locally compact hypergroup, Trans. Amer. Math. Soc. 179 (1973) 331–348.
[4] J.M.G. Fell, The dual spaces of C^*-algebras, Trans. Amer. Math. Soc. 94 (1960) 365–403.
[5] J.M.G. Fell, Weak containment and induced representations of groups II, Trans. Amer. Math. Soc. 110 (1964) 424–447.
[6] F. Filbir, R. Lasser, Reiter's condition P_2 and the Plancherel measure for hypergroups, Illinois J. Math. 44 (2000) 20–32.
[7] E.E. Granirer, Some results on $A_p(G)$ submodules of $PM_p(G)$, Colloq. Math. 51 (1987) 155–163.
[8] F.P. Greanleaf, Amenable actions of locally compact groups, J. Funct. Analysis 4 (1969) 295–315.
[9] W. Hauenschild, E. Kaniuth, A. Kumar, Harmonic analysis on central hypergroups and induced representations, Pacific J. Math. 110 (1984) 83–112.
[10] P. Hermann, Induced representations of hypergroups, Math. Z. 211 (1992) 687–699.
[11] R.I. Jewett, Spaces with an abstract convolution of measures, Adv. Math. 18 (1975) 1–101.
[12] J.P. Pier, Amenable Locally Compact Groups, Wiley & Sons, New York, 1984.
[13] H. Reiter, Classical Harmonic Analysis and Locally Compact Groups, Oxford Univ. Press, London, 1968.
[14] M. Rossler, On the dual of a commutative signed hypergroup, Manuscripta Math. 38 (1995) 147–163.
[15] M. Skantharajah, Amenable hypergroups, Illinois J. Math. 36 (1992) 15–46.
[16] H.L. Skudlarek, On a two sided version of Reiter's condition and weak containment, Arch. Math. 31 (1978) 605–610.
[17] R. Spector, Aperçu de la théorie des hypergroupes, in: Lecture Notes in Math., vol. 497, Springer, Berlin, 1975, pp. 643–673.
[18] M. Voit, On the dual space of a commutative hypergroups, Arch. Math. 56 (1991) 360–385.