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Numerical Analysis

Rapid convergence of a Galerkin projection of the KdV equation

Henrik Kalisch

NTNU, NO-7491 Trondheim, Norway

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Abstract

In this Note, it is shown that a Fourier Galerkin approximation of the Korteweg–de Vries equation with periodic boundary conditions converges exponentially fast if the initial data can be continued analytically to a strip about the real axis. ***To cite this article: H. Kalisch, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Convergence rapide d'une projection de Galerkin pour l'equation de KdV. Dans cette Note, nous montrons que l'approximation donnée par une methode de Galerkin de type Fourier de l'équation de Korteweg–de Vries avec conditions aux bords périodiques converge de façon exponentielle si les donnes initiales peuvent être prolonges analytiquement sur une bande autour de l'axe réel. ***Pour citer cet article : H. Kalisch, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

1. Introduction

We are concerned with the rate of convergence of a spectral Galerkin approximation of solutions to the Korteweg–de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0. \tag{1}$$

This equation was found by Boussinesq [2] and Korteweg and de Vries [8] as a model for the one-directional propagation of surface water waves in a narrow channel. In this context, the assumptions include the long-wavelength and small-amplitude requirements. Since its original introduction, the KdV equation has been found useful in a variety of other contexts, such as internal waves, flow in blood vessels and plasma physics, to name just a few.

Even though the equation is exactly solvable by means of the inverse scattering transform, there has been considerable interest in the numerical approximation of solutions of (1). There have been a number of successful numerical schemes for the KdV equation. An interesting review of some of these methods is given in [10]. Here we want to investigate the equation in the context of periodic boundary conditions, with a corresponding Fourier–Galerkin method.

E-mail address: kalisch@math.ntnu.no (H. Kalisch).

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In practice, a collocation method will be preferred, and this issue will be taken up in a later paper. We consider only a spatial discretization, so that the resulting semi-discrete equation is a system of ordinary differential equations.

The goal here is to improve a convergence result of Maday and Quarteroni [9], where polynomial convergence of arbitrary order for smooth data is obtained. As will be shown, if the initial data are analytic in a strip about the real axis, then the convergence rate is actually exponential. That is, if u_N denotes the Galerkin approximation, there exist positive constants Λ_T and σ_T, depending on T, such that

$$\sup_{t\in[0,T]} \left\| u(\cdot,t) - u_N(\cdot,t) \right\|_{L^2} \leqslant \Lambda_T \, \mathrm{e}^{-\sigma_T N}.$$

It should be noted that even though the result in [9] yields spectral convergence, i.e. convergence faster than any polynomial, it is not obvious that this can be used to give exponential convergence. The issue is essentially a question about whether an infinite sequence of Sobolev norms can be summed. In the context of the KdV equation, this is not a trivial question, but a positive answer was given in recent work of Bona and Grujić [1]. The convergence proof in the present article relies on their analytical result.

2. Auxiliary results

To quantify the domain of analyticity, we use the class of periodic Gevrey spaces, as introduced in [3]. Denote by $\|\cdot\|_{G_{\sigma,s}}$ the Gevrey norm given by

$$\|f\|^2_{G_{\sigma,s}} = \sum_{k\in\mathbb{Z}} \mathrm{e}^{2\sigma(1+|k|)} \left(1+|k|^2\right)^s \left|\hat{f}(k)\right|^2,$$

where the Fourier coefficients $\hat{f}(k)$ of the function f are defined by $\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}kx} f(x)\,\mathrm{d}x$. A Paley–Wiener type argument shows that functions in the space $G_{\sigma,s}$ are analytic in a strip of width 2σ about the real axis. Note that by setting σ equal to zero, we recover the usual periodic Sobolev spaces. In particular, for $\sigma = 0$ and $s = 0$, the space $L^2(0,2\pi)$ appears. In the sequel, we will also use the inner product on this space, given by $(f,g) = \int_0^{2\pi} f(x)\overline{g(x)}\,\mathrm{d}x$. The space of continuous functions from the interval $[0,T]$ into the space $G_{\sigma,s}$ is denoted by $C([0,T],G_{\sigma,s})$. The study of the KdV equation in spaces of analytic function was initiated by Kato and Masuda in [6]. The problem was subsequently studied by Hayashi [4], and more recently by Bona and Grujić in [1], where it was proved that the radius σ of spatial analyticity decreases at most exponentially over time. All these studies have been in the context of the initial-value problem on the real line. Here, we state a corresponding result for the problem on the interval $[0,2\pi]$ with periodic boundary conditions.

Existence, uniqueness and continuous dependence on the initial data of solutions to the periodic initial-value problem have been studied by Temam [11], Kenig, Ponce and Vega [7], and more recently by Kappeler and Topalov [5]. Well posedness in periodic Gevrey classes can be established in much the same way as in [1], where this was done for the initial-value problem on the real line. In particular the width of the strip in which the solution is analytic is given by

$$\sigma(t) = \sigma_0 \, \mathrm{e}^{-ct\|u_0\|_{G_{\sigma_0,s}}} \, \mathrm{e}^{-ct^{3/2}}, \tag{2}$$

for some constant c independent of t. The corresponding Gevrey norm may be estimated by

$$\left\|u(\cdot,t)\right\|_{\sigma(t),s} \leqslant \|u_0\|_{G_{\sigma_0,s}} + c\sqrt{t}, \tag{3}$$

for another constant c independent of t. The results are summarized in the following theorem.

Theorem 2.1. *Suppose that $u \in C([0,T],H^s)$ is a periodic solution of* (1) *with initial data $u_0 \in G_{\sigma_0,s}$ for some $\sigma_0 > 0$ and $s > \frac{5}{2}$. Then $u(\cdot,t)$ extends uniquely to a function in $G_{\sigma(t),s}$ with $\sigma(t)$ given by* (2). *Moreover, for any $\tau \in (0,T)$, $u \in C([0,\tau],G_{\sigma(\tau),s})$, with a bound provided by* (3).

For the initial-value problem on the real line, this theorem was proved in [1].

As is well known, the KdV equation has an infinite number of conserved integrals. Consequently, for initial data $u_0 \in G_{\sigma_0,s}$, all positive integer Sobolev norms remain bounded for all time. What is more, for $r < \frac{5}{2}$ and $s > \frac{5}{2}$, we have

$$\sup_{t\in[0,T]} \left\| u(\cdot,t) \right\|_{H^r} \leqslant \sup_{t\in[0,T]} \left\| u(\cdot,t) \right\|_{G_{\sigma(T),s}} \equiv K.$$

3. Galerkin projection

The subspace of L^2 spanned by the set $\{e^{ikx} \mid k \in \mathbb{Z},\ -N \leqslant k \leqslant N\}$ is denoted by S_N. The operator P_N denotes the orthogonal projection from L^2 onto S_N, defined by

$$P_N f(x) = \sum_{-N\leqslant k\leqslant N} e^{ikx} \hat{f}(k).$$

For $f \in G_\sigma$ and $g \in G_{\sigma,s}$, respectively, the inequalities

$$\|f - P_N f\|_{L^2} \leqslant 2\,e^{-\sigma N} \|f\|_{G_\sigma}, \qquad \|g - P_N g\|_{H^r} \leqslant 2N^{r-s}\, e^{-\sigma N} \|g\|_{G_{\sigma,s}}$$

hold for $r < s$. The proof is straightforward using the definition of P_N. Now since we have available an upper bound of the Gevrey norm for a periodic solution u of (1) with initial data $u_0 \in G_{\sigma_0,s}$ these estimates provide for

$$\left\| u(\cdot,t) - P_N u(\cdot,t) \right\|_{H^r} \leqslant 2N^{r-s}\, e^{-\sigma(t)N} \left\| u(\cdot,t) \right\|_{G_{\sigma(t),s}}, \tag{4}$$

with $\sigma(t)$ given by (2). This is the key estimate to be used in the convergence proof.

A space-discretization of (1) is defined as follows. Find a function u_N from $[0,T]$ to S_N which satisfies

$$\begin{cases} \left(\partial_t u_N + \frac{1}{2}\,\partial_x(u_N^2) + \partial_x^3 u_N, \phi\right) = 0, & t \in [0,T], \\ u_N(0) = P_N u_0, \end{cases} \tag{5}$$

for all $\phi \in S_N$. As it turns out, the discretized form of the equation also has some conserved integrals.

Lemma 3.1. *Suppose u_N is a solution of* (5)*. Then the following two equations hold.*

$$\frac{d}{dt}\int_0^{2\pi} u_N^2\, dx = 0, \qquad \frac{d}{dt}\int_0^{2\pi} \left(\partial_x u_N^2 - \frac{1}{3}u_N^3\right) dx = 0.$$

Here the dependence of u_N on x and t has been suppressed for the sake of readability. From these relations, whose proof may be found in [9], it follows immediately that the following Sobolev norms are bounded.

Corollary 3.2. *Suppose u_N is a solution of* (5)*. Then there are constants c_0 and c_1, such that*

$$\sup_{t\in[0,T]} \left\| u_N(\cdot,t) \right\|_{L^2} \leqslant c_0, \qquad \sup_{t\in[0,T]} \left\| u_N(\cdot,t) \right\|_{H^1} \leqslant c_1.$$

For a proof of this corollary, the reader is referred to [9], where the next lemma is also proved.

Lemma 3.3. *Suppose u_N is a solution of* (5)*. Then there is a constant c_2, such that*

$$\sup_{t\in[0,T]} \left\| u_N(\cdot,t) \right\|_{H^2} \leqslant c_2.$$

With these estimates in hand, we can mount an attack on proving the exponential convergence of the Galerkin scheme.

Theorem 3.4. *Suppose $u_0 \in G_{\sigma_0,s}$ for $\sigma_0 > 0$ and $s > \frac{5}{2}$. Given $T > 0$ and $N \in \mathbb{Z}_+$, there is a unique solution u_N to the finite-dimensional problem* (5)*. Moreover, there are positive constants Λ_T and σ_T, such that*

$$\sup_{t\in[0,T]} \left\| u(\cdot,t) - u_N(\cdot,t) \right\|_{L^2} \leqslant \Lambda_T N^{1-s}\, e^{-\sigma_T N}.$$

The existence of the solution u_N on the interval $[0, T]$ is proved by a combination of a fixed-point argument and the foregoing stability results. To prove the convergence estimate, consider the function $h = u_N - P_N u \in S_N$ as the test function ϕ in formula (5). Subtracting from (1), the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|h(\cdot,t)\right\|_{L^2} \leqslant \sup_t \left\|u(\cdot,t) + u_N(\cdot,t)\right\|_{H^2}\left\|h(\cdot,t)\right\|_{L^2} + 10\sup_t \left\|u(\cdot,t)\right\|^2_{G_{\sigma(T),s}} N^{1-s}\,\mathrm{e}^{-\sigma(T)N}$$

appears. Letting $\lambda = \sup_t \|u(\cdot,t)\|_{H^2} + c_2$ and using Gronwall's inequality, we gain the inequality

$$\left\|h(\cdot,t)\right\|_{L^2} \leqslant \left\|h(\cdot,0)\right\|_{L^2}\mathrm{e}^{\lambda T} + 10K^2 N^{1-s}\,\mathrm{e}^{-\sigma(T)N} T\,\mathrm{e}^{\lambda T}.$$

Noting that $\|h(\cdot,0)\|_{L^2} = 0$, and using the triangle inequality, we get the final estimate

$$\left\|u(\cdot,t) - u_N(\cdot,t)\right\|_{L^2} \leqslant \Lambda_T N^{1-s}\,\mathrm{e}^{-\sigma_T N},$$

where $\Lambda_T = 2K + 10K^2 T\,\mathrm{e}^{\lambda T}$ and $\sigma_T = \sigma(T)$ according to (2). Taking the supremum over t concludes the proof. A similar result holds for the Fourier-collocation projection of the KdV equation. In that case, the proof is somewhat more complicated, and will be given in a subsequent article.

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