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Partial Differential Equations

A mountain pass theorem without Palais–Smale condition

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Abstract

Given a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, Λ an interval of $\mathbb{R}$ and $J \in C^2(\mathcal{H}, \mathbb{R})$ whose gradient $\nabla J : \mathcal{H} \to \mathcal{H}$ is a compact mapping, we consider a family of functionals of the type:

$$I(\lambda, u) = \langle u, u\rangle - \lambda J(u), \quad (\lambda, u) \in \Lambda \times \mathcal{H}.$$

Without further compactness assumptions, we present a deformation lemma to detect critical points. In particular, if $I(\bar\lambda, \cdot)$ has a 'mountain pass structure' for some $\bar\lambda \in \Lambda$, we deduce the existence of a sequence $\lambda_n \to \bar\lambda$ for which each $I(\lambda_n, \cdot)$ has a critical point. ***To cite this article: M. Lucia, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Un théorème du col sans condition de Palais–Smale. Étant donné un espace de Hilbert $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, Λ un intervalle de $\mathbb{R}$ et $J \in C^2(\mathcal{H}, \mathbb{R})$ dont le gradient $\nabla J : \mathcal{H} \to \mathcal{H}$ est une application compacte, nous considérons une famille de fonctionelle de la forme :

$$I(\lambda, u) = \langle u, u\rangle - \lambda J(u), \quad (\lambda, u) \in \Lambda \times \mathcal{H}.$$

Sans autres hypothèses de compacité, nous présentons un lemme de déformation pour détecter des points critiques. En particulier, si $I(\bar\lambda, \cdot)$ a une structure de « col » pour un certain $\bar\lambda \in \Lambda$, nous montrons l'existence d'une suite $\lambda_n \to \bar\lambda$ pour laquelle chaque $I(\lambda_n, \cdot)$ admet un point critique. ***Pour citer cet article : M. Lucia, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Version française abrégée

Dans cette Note, nous travaillons dans un espace de Hilbert $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. Basé sur des idées qui remontent aux travaux de M. Morse, une manière de détecter les points critiques d'une fonctionelle définies dans $\mathcal{H}$, consiste

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à étudier le changement de topologie de ses ensembles de niveaux. Plus précisément, si $I \in C^2(\mathcal{H})$ satisfait la *condition de Palais–Smale*, alors un lemme de déformation permet de conclure l'alternative suivante :

(1) soit $\{u \in \mathcal{H}\colon\ I(u) \leqslant a\}$ et $\{u \in \mathcal{H}\colon\ I(u) \leqslant b\}$ sont topologiquement équivalents,
(2) soit I admet un point critique dans $\{u \in \mathcal{H}\colon\ a \leqslant I(u) \leqslant b\}$.

Lorsque la fonctionelle I admet une géométrie dite de « col », Ambrosetti et Rabinowitz notèrent que la première alternative est impossible pour certains niveaux a, b (voir [1]). Ceci a donné lieu au fameux « Théorème du col » qui est à la fois simple et puissant pour démontrer l'existence de points critiques. Mais dans de nombreux problèmes importants, l'hypothèse de compacité de Palais–Smale s'est avérée une restriction sérieuse pour appliquer ce théorème et dès lors divers travaux ont été entrepris afin d'y remédier. Un exemple illustratif et d'importance dans divers problèmes de géométrie et de physique est donné par la fonctionelle (1), où M est une variété compacte de dimension 2 et de volume 1. Les arguments de Struwe et Tarantello [4] montrent que cette fonctionelle a une structure de « col » aussitôt que $\lambda_1(M) > 8\pi$ et $\lambda \in (8\pi, \lambda_1(M))$, où $\lambda_1(M)$ désigne la première valeur propre positive du Laplacien de la variété. Vu que dans ce cas la condition de Palais–Smale n'est pas bien comprise, la difficulté est surmontée en utilisant la monotonicité en λ de la fonctionelle. Leur méthode permet alors de construire un point critique non-trivial pour une famille $I(\lambda, \cdot)$ où λ appartient à un sous-ensemble dense dans l'intervalle $(8\pi, \lambda_1(M))$. Une approche similaire a été aussi utilisée dans [3].

Cette Note a pour but de montrer qu'une telle conclusion peut être obtenue sans faire usage de la monotonicité et s'inscrit dans un cadre abstrait bien plus général. Considérons un intervalle ouvert $\Lambda \subset \mathbb{R}$ et une famille de fonctionelle satisfaisant (2) and (3) (l'inégalité de Moser montre que (1) a une telle structure). Si $I(\bar\lambda, \cdot)$ a une structure de « col » pour un certain $\bar\lambda \in \Lambda$, nous montrons alors l'existence d'une suite $\lambda_n \to \bar\lambda$ pour laquelle chaque $I(\lambda_n, \cdot)$ admet un point critique. Dans un travail ultérieur nous ferons une analyse plus détaillée et donnerons divers exemples où ce résultat s'applique.

1. Introduction

In this Note, we shall work in a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ and denote the associated norm by $|\cdot|$. Based on some ideas that go back to the works of M. Morse, a way to detect critical points of a functional defined in $\mathcal{H}$ consists in studying the change of topology of its level sets. More precisely, if $I \in C^2(\mathcal{H})$ satisfies the so-called *Palais–Smale condition* ((PS)-condition for short), then a classical deformation lemma yields the following alternative:

(1) either $\{u \in \mathcal{H}\colon\ I(u) \leqslant a\}$ and $\{u \in \mathcal{H}\colon\ I(u) \leqslant b\}$ are topologically equivalent,
(2) or I has a critical point in the set $\{u \in \mathcal{H}\colon\ a \leqslant I(u) \leqslant b\}$.

When the functional I exhibits a so-called 'mountain pass geometry', Ambrosetti et Rabinowitz noted that the first alternative cannot hold for some value of a, b (see [1]). This is the main point of the famous 'mountain pass Theorem' that is simple as well as powerful for proving the existence of critical points. However, in several important problems, the compactness assumption of Palais–Smale could be a serious restriction to apply this theorem. Therefore several works have been undertaken to overcome such a difficulty. A typical example of relevance in several problems of geometry and physics is given by the following functional:

$$I(\lambda, u) := \frac{1}{2}\int_M |\nabla u|^2 - \lambda \log\left(\int_M \mathrm{e}^u\right) + \lambda \int_M u, \quad u \in H^1(M), \tag{1}$$

where M is a compact manifold of dimension 2 and volume 1. The arguments of Struwe–Tarantello in [4] show that this functional exhibits a 'mountain pass geometry' whenever $\lambda_1(M) > 8\pi$ and $\lambda \in (8\pi, \lambda_1(M))$, where $\lambda_1(M)$ denotes the first positive eigenvalue of the Laplacian in M. Since in this case the (PS)-condition is not well-understood, in [4] the difficulty is overcome by exploiting the monotonicity of the functional in the variable λ.

Their method allows then to construct a nontrivial critical point for a family $I(\lambda,\cdot)$ where λ belongs to a dense subset of the interval $(8\pi,\lambda_1(M))$. A similar approach has been used in [3].

The aim of this Note is to show that such a conclusion can be obtained without any use of monotonicity and can be viewed in a much general framework. Indeed, let $\Lambda\subset\mathbb{R}$ be an open interval and consider a family of functional of the type:

$$I(\lambda,u)=\langle u,u\rangle-\lambda J(u),\quad (\lambda,u)\in\Lambda\times\mathcal{H}, \tag{2}$$

$$J\in C^2(\mathcal{H})\quad\text{and}\quad \nabla J:\mathcal{H}\to\mathcal{H}\quad\text{is compact}, \tag{3}$$

(the Moser's inequality shows that (1) has such a structure, see [2]). Our main result can be stated as follows:

Theorem 1.1. *Let $I(\lambda,\cdot)$ satisfy* (2), (3). *Assume that for some $I(\bar\lambda,\cdot)$, there exist $h_0,h_1\in\mathcal{H}$ and $\rho_0>0$ with the properties*:

$$\begin{cases} |h_1-h_0|>\rho_0,\\ \bar\alpha:=\max\{I(\bar\lambda,h_0),I(\bar\lambda,h_1)\}<\bar\beta:=\inf\limits_{|u-h_0|=\rho_0}\{I(\bar\lambda,u)\}.\end{cases} \tag{4}$$

By setting $\Gamma:=\{\gamma\in C([0,1],\mathcal{H}):\ \gamma(0)=h_0,\ \gamma(1)=h_1\}$, let us define:

$$\bar c:=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\{I(\bar\lambda,\gamma(t))\}\quad(\geqslant\bar\beta). \tag{5}$$

Then, for each $\varepsilon\in(0,\bar c-\bar\alpha)$, there exists a sequence $(\lambda_n,u_n)\in\Lambda\times\mathcal{H}$ satisfying

$$\begin{cases} D_uI(\lambda_n,u_n)=0,\quad \lambda_n\in[0,\lambda),\quad \lambda_n\to\lambda,\\ \bar c-\epsilon<I(\lambda_n,u_n)<\bar c+\epsilon.\end{cases} \tag{6}$$

In a coming work, we will give a more detail analysis and treat several examples where Theorem 1.1 can be applied.

2. A deformation lemma

Definition 2.1. Given two sets $A\subset B\subset\mathcal{H}$, we say that A is a deformation retract of B if there exists a continuous map $\eta:[0,1]\times\mathcal{H}\to\mathcal{H}$ satisfying

$$\eta(t,u_0)=u_0\quad\forall(t,u_0)\in[0,1]\times A\quad\text{and}\quad\eta(1,\cdot)\text{ maps }B\text{ on }A. \tag{7}$$

The following lemma can be derived easily:

Lemma 2.2. *Let $(X_n,Y_n)\in\mathcal{H}\times\mathcal{H}$ and set $Z_n:=-\{(1+|Y_n|)X_n+|X_n|Y_n\}$. Then, $\langle X_n,Z_n\rangle\leqslant0$. Furthermore, if $\lim_{n\to\infty}\langle X_n,Z_n\rangle=0$, then $\lim_{n\to\infty}\frac{Z_n}{1+|Y_n|}=0$.*

The deformation lemma we will need can be stated as follows:

Proposition 2.3. *Let $I(\lambda,\cdot)$ satisfy* (2) *and* (3). *Given $\bar\lambda\in\Lambda$ and $a,b\in\mathbb{R}$ $(a<b)$, the following alternative holds*:

(1) *either there exists a sequence $(\lambda_n,u_n)\in\Lambda\times\mathcal{H}$ satisfying*:

$$\begin{cases} u_n\text{ is a critical point of }I(\lambda_n,\cdot),\\ a\leqslant I(\bar\lambda,u_n)\leqslant b,\quad \lambda_n\in(0,\bar\lambda],\quad \lambda_n\to\bar\lambda;\end{cases} \tag{8}$$

(2) *or, $\{u:\ I(\bar\lambda,u)\leqslant a\}$ is a deformation retract of $\{u:\ I(\bar\lambda,u)\leqslant b\}$.*

Proof. We write $\bar{I}(u)$ instead of $I(\bar{\lambda}, u)$ and introduce the following sets:

$$\bar{I}^a := \{u \in \mathcal{H}\colon\ \bar{I}(u) \leqslant a\}, \quad \bar{I}_a^b := \{u \in \mathcal{H}\colon\ a \leqslant \bar{I}(u) \leqslant b\} \quad (a, b \in \mathbb{R}).$$

By considering $-J$ instead of J, we may assume that $\bar{\lambda} > 0$. In the case (8) does not hold, we can find $\epsilon > 0$ such that

$$D_u I(\lambda, u) \neq 0, \quad \forall(\lambda, u) \in [\bar{\lambda} - \epsilon, \bar{\lambda}] \times \bar{I}_a^b. \tag{9}$$

Under this assumption, we construct a flow which deforms $\bar{I}^b$ on $\bar{I}^a$ by keeping J bounded along the flow-line. To do this let $Z \in C^1(\mathcal{H}, \mathcal{H})$ be defined by:

$$Z(u) := -\{(1 + |\nabla J(u)|)\nabla \bar{I}(u) + |\nabla \bar{I}(u)|\nabla J(u)\}, \tag{10}$$

and $\omega_\varepsilon \in \mathcal{C}^\infty(\mathbb{R})$ be such that

$$0 \leqslant \omega_\varepsilon \leqslant 1, \qquad \omega_\varepsilon(\zeta) = 0 \quad \forall\zeta \leqslant \varepsilon, \qquad \omega_\varepsilon(\zeta) = 1 \quad \forall\zeta \geqslant 2\varepsilon.$$

Consider then the local flow $\eta = \eta(t, u_0)$ defined by the Cauchy problem:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\omega_\varepsilon\left(\frac{|\nabla \bar{I}(u)|}{1 + |\nabla J(u)|}\right)\nabla \bar{I}(u) + Z(u), \quad u(0) = u_0. \tag{11}$$

Using Lemma 2.2, we can see that $\frac{\mathrm{d}}{\mathrm{d}t}[\bar{I} \circ \eta(\cdot, u_0)](t) \leqslant 0$, namely $\bar{I}$ decreases along the flow-line. We will actually prove that given $u_0 \in \bar{I}^b$, we have $\eta(t, u_0) \in \bar{I}^a$ at some $t = t(u_0)$. To prove this fact, we are led to study the quantities $J(\eta(t, u_0))$ and $|\eta(t, u_0)|$ when $\eta(t, u_0) \in \bar{I}_a^b$. It is first possible to derive

$$\frac{\mathrm{d}}{\mathrm{d}t}[J \circ \eta(\cdot, u_0)](t) \leqslant -\frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}[\bar{I} \circ \eta(\cdot, u_0)](t).$$

Therefore after integration, we obtain:

$$J(\eta(t, u_0)) < J(u_0) + \frac{b - a}{\epsilon} \quad \text{whenever } \eta(t, u_0) \in \bar{I}_a^b. \tag{12}$$

Hence, whenever $\eta(t, u_0)$ belongs to $\bar{I}_a^b$, we see by using (12) and $\bar{\lambda} > 0$ that

$$|\eta(t, u_0)|^2 \leqslant \bar{\lambda} J(\eta(t, u_0)) + b \leqslant C(a, b, u_0, \epsilon). \tag{13}$$

We claim that

$$\frac{\mathrm{d}}{\mathrm{d}t}[\bar{I} \circ \eta(\cdot, u_0)] \leqslant -c^2 < 0 \quad \text{whenever } \eta(t, u_0) \in \bar{I}_a^b. \tag{14}$$

Indeed, assume the existence of a sequence $t_n \geqslant 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}[\bar{I} \circ \eta(\cdot, u_0)](t_n) \to 0, \quad \eta(t_n, u_0) \in \bar{I}_a^b. \tag{15}$$

We set $u_n := \eta(t_n, u_0)$. By (13) together with the assumption (3), we derive

$$u_n \rightharpoonup \tilde{u} \quad \text{weakly in } \mathcal{H} \quad \text{and} \quad \nabla J(u_n) \to \nabla J(\tilde{u}) \quad \text{strongly in } \mathcal{H}. \tag{16}$$

On the other hand, from (15) and the fact that $\langle \nabla I(u_n), Z(u_n)\rangle \leqslant 0$ (Lemma 2.2), we deduce that

$$\omega_\epsilon\left(\frac{|\nabla I(u_n)|}{1 + |\nabla J(u_n)|}\right)|\nabla I(u_n)|^2 \to 0 \quad \text{and} \quad \langle \nabla I(u_n), Z(u_n)\rangle \to 0.$$

Consequently, we must have

$$\frac{|\nabla I(u_n)|}{1 + |\nabla J(u_n)|} \to \gamma \leqslant \epsilon \quad \text{and} \quad \langle \nabla I(u_n), Z(u_n)\rangle \to 0. \tag{17}$$

From Lemma 2.2, we deduce

$$\nabla I(u_n) + \frac{|\nabla I(u_n)|}{1+|\nabla J(u_n)|}\nabla J(u_n) \to 0.$$

Therefore, for all $\varphi \in \mathcal{H}$, we have

$$\langle u_n, \varphi\rangle - \left\{\bar{\lambda} - \frac{|\nabla I(u_n)|}{1+|\nabla J(u_n)|}\right\}\langle \nabla J(u_n), \varphi\rangle \to 0. \tag{18}$$

By setting $\gamma_n := \frac{|\nabla I(u_n)|}{1+|\nabla J(u_n)|}$ and by using (17), (16) and (18), we derive

$$\langle \tilde{u}, \varphi\rangle - (\bar{\lambda} - \gamma)\langle \nabla J(\tilde{u}), \varphi\rangle = 0 \quad (\gamma \leqslant \varepsilon). \tag{19}$$

Moreover, by choosing $u_n - \tilde{u}$ as a test function in (18), we get

$$|u_n - \tilde{u}|^2 + \langle \tilde{u}, u_n - \tilde{u}\rangle - (\bar{\lambda} - \gamma_n)\langle \nabla J(u_n), u_n - \tilde{u}\rangle \to 0.$$

Therefore, using (16), we derive that $|u_n - \tilde{u}| \to 0$, implying $\tilde{u} \in \bar{I}_a^b$. This conclusion together with (19) contradict our initial assumption (9). Hence, (15) is impossible and so (14) must hold. Consequently, whenever $\eta(t, u_0) \in \bar{I}_a^b$ we have $I(\eta(t, u_0)) \leqslant -c^2 t + I(u_0)$. Hence, there is a t such that $\eta(t, u_0) \in \bar{I}^a$. Now, classical arguments show that $\bar{I}^a$ is a deformation retract of $\bar{I}^b$. □

Proof of Theorem 1.1. If there is no sequence (λ_n, u_n) satisfying (4), the deformation lemma as stated in Proposition 2.3 shows that $\{u\colon\ I(\bar{\lambda}, u) \leqslant \bar{c} - \epsilon\}$ is a deformation retract of $\{u\colon\ I(\bar{\lambda}, u) \leqslant \bar{c} + \epsilon\}$. Similar arguments as in [1] give a contradiction. □

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