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Harmonic Analysis

An extremal problem for Fourier transforms of probabilities

Shunlong Luo [a], Zhengmin Zhang [b]

[a] *Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing 100080, PR China*
[b] *School of Mathematics and Statistics, Carleton University, Ottawa, ON K1S 5B6, Canada*

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Abstract

We study an extremal problem concerning the supremum of the Fourier transforms (characteristic functions) of probability distributions under the constraint that the Fourier transforms vanish at a fixed point. This problem arises from the investigation of the survival amplitudes of quantum states driven by Schrödinger dynamics, and has general and curious implications for the evolution pictures of quantum systems. ***To cite this article: S. Luo, Z. Zhang, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Un problème extrêmal pour transformée de Fourier de probabilité. On résout un problème extrêmal concernant des fonctions caractéristiques soumises à la condition de s'annuler en un point fixé. L'origine du problème est l'étude de l'amplitude de survie d'un état quantique dans la dynamique de Schrödinger, et la solution exprime un phénomène curieux dans l'évolution des systèmes quantiques. ***Pour citer cet article : S. Luo, Z. Zhang, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

1. Introduction

For any probability distribution function F (that is, F is a non-decreasing, right continuous function on R, with $\lim_{t\to-\infty} F(t)=0$, $\lim_{t\to\infty} F(t)=1$), let $\phi(t)=\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tx}\,\mathrm{d}F(x)$ be its Fourier transform. In probability theory, ϕ is usually called the characteristic function of F [4]. Let Φ be the set of all characteristic functions, and for any fixed $T>0$, let $\Phi_T \equiv \{\phi\in\Phi\colon \phi(T)=0\}$ be the set of all characteristic functions which vanish at $t=T$. We want to address the following, physically motivated, problem:

For $t\in[0,T]$, what is the supremum of $|\phi(t)|$ when ϕ varies in Φ_T? That is, we want to determine the function $M_T(t)$ defined by the extremal problem $M_T(t)\equiv \sup_{\phi\in\Phi_T}|\phi(t)|$, $t\in[0,T]$.

In quantum mechanics, $\phi(t)$ may be interpreted as the survival amplitude (whose absolute square is the survival probability or decay law) of a quantum state with energy distribution $\mathrm{d}F(x)$, and is useful in characterizing the

E-mail addresses: luosl@amt.ac.cn (S. Luo), zhangzm@math.carleton.ca (Z. Zhang).

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evolution speed of quantum states and time-energy uncertainty relations [3,5,6]. The condition $\Phi(T)=0$ means that the quantum state evolves into an orthogonal state at time T. It is natural to ask what possible values can the survival amplitude take at some earlier time t.

Before we attack the above problem, let us first consider some examples in order to appreciate the intricacy of this problem.

For any positive, relatively prime integers m and n satisfying $2 \leqslant m < n$, put $d = \frac{n}{m}\frac{2\pi}{T}$, and consider the characteristic function $\phi(t)$ of a uniform lattice distribution supported on $\{-jd\colon\ j=1,2,\ldots,m\}$:

$$\phi(t)=\frac{1}{m}\sum_{j=1}^{m}\mathrm{e}^{\mathrm{i}tjd}.$$

Clearly, $\phi(T)=0$ and $|\phi(2\pi/d)|=1$, that is $|\phi(t)|=1$ when $t=\frac{m}{n}T$. Consequently, we know that $M_T(t)=1$ for any $t\in\{\frac{m}{n}T\colon\ 2\leqslant m<n,\ m \text{ and } n \text{ are relatively prime}\}$. The above set consists of all rational multiples of T in $(0,T)$ except those with numerator 1.

Based on the above example, it is tempting to guess that $M_T(t)=1$ for any $t\in[0,T)$ since it seems that we have such a large freedom to vary the characteristic functions. However, there are some curious exceptions which are of number-theoretic origin. In fact, by use of the inequality (see Feller [1], page 527)

$$\left|\phi(t)\right|^2\leqslant\frac{1}{2}\left(1+\left|\phi(2t)\right|\right)$$

which holds for any characteristic function $\phi(t)$, and by considering $\phi(t)=\cos\frac{\pi t}{2T}$, we readily conclude that

$$M_T\left(\frac{T}{2^k}\right)=\cos\frac{\pi}{2^{k+1}},\quad k=1,2,\ldots.$$

Our main result may be stated as follows.

Theorem 1.1. *Put $\frac{T}{t}=\omega$, then $M_T(t)=\cos\frac{\pi}{2\omega}$ if ω is an integer, and $M_T(t)=1$ otherwise.*

It is interesting to compare our problem with a similar one proposed by Fryntov [2]. Specifically, let $\Phi_{T,0}\subset\Phi_T$ be the subset of Φ_T consisting of all characteristic functions which vanish on the interval $[T,\infty)$. Let $M_{T,0}(t)\equiv\sup_{\phi\in\Phi_{T,0}}|\phi(t)|$, $t\in(0,T)$. Then apparently, $M_{T,0}(t)\leqslant M_T(t)$. Fryntov proved that

$$M_{T,0}(t)=\cos\frac{\pi}{n+1},\quad \forall t\in\left[\frac{T}{n},\frac{T}{n-1}\right),\ n=2,3,\ldots.$$

2. Determination of $M_T(t)$ when $\frac{T}{t}$ is an integer

In this section, we show that $M_T(t)=\cos\frac{\pi}{2\omega}$ when $\frac{T}{t}\equiv\omega$ is an integer. This is a consequence of the following elementary facts.

Lemma 2.1. *Let $\Phi_T^R\subset\Phi_T$ be the set consisting of all real-valued characteristic functions $\phi(t)$ such that $\phi(T)=0$. Then for any $t\in(0,T)$,*

$$M_T(t)=\sup_{\phi\in\Phi_T^R}\phi(t).$$

Proof. Clearly,

$$M_T(t)\equiv\sup_{\phi\in\Phi_T}\left|\phi(t)\right|\geqslant\sup_{\phi\in\Phi_T^R}\phi(t).$$

On the other hand, for any characteristic function $\phi(t)$, and for any $\gamma \in R$, the function $\mathrm{Re}(\mathrm{e}^{\mathrm{i}\gamma t}\phi(t))$ (real part) is a real-valued characteristic function, and moreover, for any fixed t, there always exists γ (which may depend on t) such that $\mathrm{e}^{\mathrm{i}\gamma t}\phi(t)$ is real and non-negative, and this in turn implies that

$$M_T(t) \equiv \sup_{\phi\in\Phi_T} \left|\phi(t)\right| \leqslant \sup_{\phi\in\Phi_T^R} \phi(t). \qquad \square$$

Lemma 2.2. *For any $t \in R$ and any positive integer n, and for any real-valued characteristic function $\phi(t)$, it holds that*

$$\phi(t) \leqslant \phi(nt)\frac{1}{n}\sin\frac{\pi}{2n} + \cos\frac{\pi}{2n}.$$

Moreover, for any fixed $T > 0$, the above inequality can become an equality at $t = \frac{T}{n}$ by taking $\phi(t) = \cos\frac{\pi t}{2T}$ which satisfies $\phi(T) = 0$.

Proof. The desired inequality follows from integrating the following inequality with respect to $\mathrm{d}F$ (the probability distribution of $\phi(t)$):

$$\cos t - \frac{1}{n}\sin\frac{\pi}{2n}\cos nt \leqslant \cos\frac{\pi}{2n}, \quad t \in R.$$

To prove this latter inequality, put $f(t) = \cos t - \frac{1}{n}\sin\frac{\pi}{2n}\cos nt$. For fixed n, we want to find the maximum value of $f(t)$ when t changes in R. Clearly, $f(t)$ is a bounded, smooth function with period 2π. It suffices to restrict t to $[0, 2\pi]$. The maximum value of f can only occur at stationary points (that is, the zero points of $f'(t)$) of f or at the end points $t = 0, 2\pi$. Now by putting $f'(t) = -\sin t + \sin\frac{\pi}{2n}\sin nt = 0$, we obtain the solutions $t = 0, \pi, 2\pi, \frac{\pi}{2n}, 2\pi - \frac{\pi}{2n}$ for n even, and $t = 0, \pi, 2\pi, \frac{\pi}{2n}, \pi - \frac{\pi}{2n}, \pi + \frac{\pi}{2n}, 2\pi - \frac{\pi}{2n}$ for n odd. Evaluating $f(t)$ at these points, one readily concludes that $\max_{t\in R} f(t) = \cos\frac{\pi}{2n}$. $\square$

3. Determination of $M_T(t)$ when $\frac{T}{t}$ is not an integer

For any $\frac{\pi}{2} \leqslant \theta \leqslant \frac{3\pi}{2}$ and integer $n \geqslant 1$, put

$$p_1 = p_3 = \frac{1}{2}\frac{1}{1-\cos\theta}, \quad p_2 = \frac{-\cos\theta}{1-\cos\theta}, \quad d = \frac{2n\pi + \theta}{T}.$$

Then clearly $\{p_1, p_2, p_3\}$ is a probability vector. Let $\phi(t) = \sum_{j=1}^{3} p_j \mathrm{e}^{\mathrm{i}tjd}$ be the characteristic function of the lattice distribution supported on $\{-jd\colon\ j = 1, 2, 3\}$ with probabilities $\{p_j\colon\ j = 1, 2, 3\}$. It is easily verified that $\phi(T) = 0$ and $|\phi(t)| = 1$ for any $t = \frac{2\pi m}{2\pi n+\theta}T,\ m = 1, 2, \ldots, n$. Noting $\frac{\pi}{2} \leqslant \theta \leqslant \frac{3\pi}{2}$, we conclude that $M_T(t) = 1$ when

$$t \in \bigcup_{n\geqslant 1,\, 1\leqslant m\leqslant n} \left[\frac{4m}{4n+3}T, \frac{4m}{4n+1}T\right].$$

We want to show that

$$[0, T]\Big/\left\{\frac{T}{n}\colon\ n = 1, 2, 3, \ldots\right\} = \bigcup_{n\geqslant 1,\, 1\leqslant m\leqslant n} \left[\frac{4m}{4n+3}T, \frac{4m}{4n+1}T\right].$$

In fact, for any relatively prime integers p and q satisfying $2 \leqslant p < q$, we can find a positive integer k such that $p < 4k < 3p$, and can also find positive integers $m' < n'$ such that $m'q - n'p = 1$. Therefore, multiplying both sides of $m'q - n'p = 1$ by k and put $m = m'k,\ n = n'k$, we have $mq = np + k$ which implies $(4n+1)p < 4mq < (4n+3)p$. That is,

$$\frac{p}{q}T \in \left[\frac{4m}{4n+3}T, \frac{4m}{4n+1}T\right].$$

In summary, we have shown that any rational multiple $\frac{p}{q}T$ of T with $2 \leqslant p < q$ belongs to an interval of the form $[\frac{4m}{4n+3}T, \frac{4m}{4n+1}T]$. However, the set consisting of all such rational multiples is dense in $[0, T]$, we conclude that the set $[0, T]/\{\frac{T}{n}\colon n = 1, 2, 3, \ldots\}$ and the set $\bigcup_{n\geqslant 1, 1\leqslant m\leqslant n}[\frac{4m}{4n+3}T, \frac{4m}{4n+1}T]$ are identical. This in turn implies that $M_T(t) = 1$ for any $t \in [0, T]/\{\frac{T}{n}\colon n = 1, 2, 3, \ldots\}$.

4. Implications for quantum evolutions

The characteristic function of a probability distribution has a physical interpretation as the survival amplitude (whose absolute square is the survival probability) of a quantum state. Following the physicist's terminology and considering the evolution of an arbitrary initial quantum state $|\psi\rangle$ (represented by a normalized wave function) driven by a time-independent energy observable (Hamiltonian) H, the evolving state $|\psi_t\rangle$ is determined by the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}|\psi_t\rangle = H|\psi_t\rangle, \qquad |\psi_0\rangle = |\psi\rangle,$$

where $\hbar$ is the Planck constant divided by 2π. Formally, the solution is given by $|\psi_t\rangle = \mathrm{e}^{-\mathrm{i}tH/\hbar}|\psi\rangle$, and the survival amplitude at time t is defined as $\phi(t) = \langle\psi|\psi_t\rangle = \langle\psi|\mathrm{e}^{-\mathrm{i}tH/\hbar}|\psi\rangle$, $t \in R$.

Now let $\{|E\rangle\}$ be the complete set of the energy eigenstates:

$$H|E\rangle = E|E\rangle, \qquad \langle E'|E\rangle = \delta(E' - E).$$

Let $|\psi\rangle$ be expanded in the energy eigenstates as $|\psi\rangle = \int \lambda(E)|E\rangle\,\mathrm{d}E$, where the integration (and also all subsequent integrations) is over the spectrum of H. When the energy spectrum is discrete, all integrals should be interpreted as discrete sums. Then $\mathrm{e}^{-\mathrm{i}tH/\hbar}|\psi\rangle = \int \mathrm{e}^{-\mathrm{i}tE/\hbar}\lambda(E)|E\rangle\,\mathrm{d}E$, and by the Parseval theorem,

$$\phi(t) = \langle\psi|\mathrm{e}^{-\mathrm{i}tH/\hbar}|\psi\rangle = \int \mathrm{e}^{-\mathrm{i}tE/\hbar}\left|\lambda(E)\right|^2\mathrm{d}E.$$

Consequently, the survival amplitude is precisely the characteristic function of the state probability density $|\lambda(E)|^2$ in energy representation if we replace t by $t/\hbar$. The condition $\phi(T) = 0$ means that the initial state $|\psi\rangle$ evolves into an orthogonal state at $t = T$. Our result indicates then a restriction on the values of $\phi(t)$ for other t. For example, if the state $|\psi\rangle$ will die at $t = T$ (that is, $\phi(T) = 0$), then it can never revive (that is, $\phi(t) = 1$) at $t = \frac{T}{n}$ for any integer n, although it may revive at other instants.

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