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Numerical Analysis/Partial Differential Equations

A two-grid approximation scheme for nonlinear Schrödinger equations: dispersive properties and convergence

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Abstract

We introduce a two-grid finite difference approximation scheme for the free Schrödinger equation. This scheme is shown to converge and to posses appropriate dispersive properties as the mesh-size tends to zero. A careful analysis of the Fourier symbol shows that this occurs because the two-grid algorithm (consisting in projecting slowly oscillating data into a fine grid) acts, to some extent, as a filtering one. We show that this scheme converges also in a class of nonlinear Schrödinger equations whose well-posedness analysis requires the so-called Strichartz estimates. This method provides an alternative to the method introduced by the authors [L.I. Ignat, E. Zuazua, Dispersive properties of a viscous numerical scheme for the Schrödinger equation, C. R. Math. Acad. Sci. Paris 340 (7) (2005) 529–534] using numerical viscosity. ***To cite this article: L.I. Ignat, E. Zuazua, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Un schéma de discrétisation bi-maille pour les équations de Schrödinger non-linéaires : propriétés dispersives et convergence. On introduit une méthode bi-maille semi-discrète en différences finies pour l'approximation numérique de l'équation de Schrödinger. On démontre la convergence L^2 du schéma et des propriétés dispersives uniformes par rapport au pas du maillage. Une analyse soigneuse en Fourier du symbole du schéma (consistant essentiellement à projeter des données lentes sur un maillage fin) montre que l'algorithme bi-maille agit comme un filtre des hautes fréquences. On montre aussi la convergence du schéma dans une classe d'équations non-linéaires dont l'étude dans le cas continu nécessite des inégalités de Strichartz. Cette méthode donne une approche alternative à celle introduite par les auteurs [L.I. Ignat, E. Zuazua, Dispersive properties of a viscous numerical scheme for the Schrödinger equation, C. R. Math. Acad. Sci. Paris 340 (7) (2005) 529–534] à l'aide d'un schéma avec viscosité numérique. ***Pour citer cet article : L.I. Ignat, E. Zuazua, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

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On introduit un schéma bi-maille en différences finies pour l'approximation numérique de l'équation de Schrödinger. Ce travail est motivé par [3] où l'on a constaté que le schéma conservatif classique semi-discret en différences finies ne possède par les propriétés dispersives uniformes (par rapport au pas du maillage), nécessaires pour garantir la convergence pour des équations non-linéaires. Dans [3] on avait introduit un schéma dissipatif, incluant un terme de viscosité numérique et on avait démontré la convergence du schéma. La viscosité numérique avait comme but de dissiper les hautes fréquences numériques qui étaient la cause du manque de dispersion.

Dans cette Note on introduit une méthode bi-maille qui, étant de caractère conservatif, joue le même rôle en filtrant les hautes fréquences. Dans le cas de l'équation de Schrödinger libre la méthode consiste simplement a projeter les données lentes d'un maillage épais dans un maillage fin, le rapport entre les deux mailles étant convenablement choisi : $1/4$. Cette méthode est inspirée de celle introduite par R. Glowinski dans [2] pour l'approximation numérique des contrôles frontière de l'équation des ondes, dont l'efficacité est maintenant bien établie, la convergence ayant été démontrée récemment dans [6].

Dans le cas linéaire, moyennant une analyse de Fourier, et en utilisant des arguments analogues à ceux de la théorie continue [1,4,5,8], on montre des propriétés dispersives (inégalités de Strichartz) uniformes par rapport au pas du maillage, la convergence L^2 de la méthode étant standard (consistance + stabilité).

Dans le cas non-linéaire on doit introduire une discrétisation soigneuse de la non-linearité, permettant de garder ses propriétés de cancellation au niveau des estimations d'energie, et qui ne fournit qu'une source d'oscillations lentes, admissibles dans notre schéma bi-maille. Ceci étant fait, on démontre la convergence du schéma dans le cas non-linéaire.

1. Introduction

Let us consider the 1-d linear Schrödinger Equation (LSE) in the whole line

$$\mathrm{i}u_t + u_{xx} = 0, \quad x \in \mathbb{R},\ t > 0; \qquad u(0,x) = \varphi(x), \quad x \in \mathbb{R}. \tag{1}$$

Its solution is given by $u(t) = S(t)\varphi$, where $S(t) = \mathrm{e}^{\mathrm{i}t\Delta}$ is the free Schrödinger operator which defines a unitary transformation group in $L^2(\mathbb{R})$. The conservation of the L^2-norm $\|u\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})}$, together with the classical estimate $|u(t,x)| \leqslant (4\pi|t|)^{-1/2}\|\varphi\|_{L^1(\mathbb{R})}$, leads by interpolation to the following $L^{p'} - L^p$ result: $\|S(t)\varphi\|_{L^p(\mathbb{R})} \leqslant c(p)t^{-1/2(1/p'-1/p)}\|\varphi\|_{L^{p'}(\mathbb{R})}$, for all $p \geqslant 2$ and $t \neq 0$. More refined space–time estimates known as the *Strichartz inequalities* show that, in addition to the decay of the solution as $t \to \infty$, the linear semigroup $S(t)$ satisfies $\|S(\cdot)\varphi\|_{L^q(\mathbb{R},L^r(\mathbb{R}))} \leqslant C\|\varphi\|_{L^2(\mathbb{R})}$ for suitable values of q and r, the so-called admissible pairs satisfying $2/q = 1/2 - 1/r$. Also a local gain of $1/2$ space derivate occurs in $L^2_{x,t}$. These properties are not only relevant for a better understanding of the dynamics of the linear system but also to derive well-posedness results for the nonlinear Schrödinger equations [1,8].

In this Note we introduce a two-grid finite-difference semi-discretization scheme that reproduces these properties, uniformly with respect to the mesh-size. Let us first consider the finite-difference conservative numerical scheme

$$\mathrm{i}u^h_t + \Delta_h u^h = 0, \quad t \in \mathbb{R}; \qquad u^h(0) = \varphi^h. \tag{2}$$

Here u^h stands for the infinite vector unknown $\{u^h_j\}_{j\in\mathbb{Z}}$, $u_j(t)$ being the approximation of the solution at the node $x_j = jh$, and Δ_h the classical second order finite difference approximation of ∂^2_x: $(\Delta_h u)_j = (u_{j+1} - 2u_j + u_{j-1})/h^2$.

This scheme satisfies the classical properties of consistency and stability which imply the L^2 convergence. The same convergence results hold for semilinear equations $\mathrm{i}u_t + u_{xx} = f(u)$ (NSE) provided that the nonlinearity f

is globally Lipshitz. However, as proved in [8] (see also [1]), the NSE is also well-posed for some nonlinearities that grow superlinearly at infinity. This well-posedness result may not be proved simply as a consequence of the L^2 conservation property. Indeed, the dispersive properties of the LSE play a crucial role. Accordingly, one may not expect to prove convergence of the numerical scheme in this class of nonlinearities without similar dispersive estimates, that should be uniform on the mesh-size parameter $h \to 0$. As we proved in [3] this conservative scheme (2) fails to have uniform dispersive properties. This is due to the high frequency spurious numerical solutions the scheme (2) introduces.

We remark that there are *slight* but important differences between the symbols of the operators $-\Delta$ and $-\Delta_h$: $p(\xi) = \xi^2,\ \xi \in \mathbb{R}$ for $-\Delta$ and $p_h(\xi) = 4/h^2 \sin^2(\xi h/2),\ \xi \in [\pi/h, \pi/h]$ for $-\Delta_h$. The symbol $p_h(\xi)$ changes convexity at the points $\xi = \pm\pi/2h$ and has critical points also at $\xi = \pm\pi/h$, two properties that the continuous symbol does not fulfill because of its strict convexity.

To compensate this lack of dispersion we propose a two-grid algorithm (inspired in [2]) and that, to some extent, acts as a filter for those unwanted high frequency components. This method is a natural alternative to the one introduced in [3], based on the use of numerical viscosity.

The method is roughly as follows. We consider two meshes: the coarse one of size $4h$, $h > 0$, $4h\mathbb{Z}$, and the fine one, $h\mathbb{Z}$, of size $h > 0$. The method relies basically on solving the finite-difference semi-discretization (2) on the fine mesh $h\mathbb{Z}$, but only for slow data, interpolated from the coarse grid $4h\mathbb{Z}$. This particular structure of the data cancels the two pathologies of the discrete symbol mentioned above and suffices to recover the dispersive properties of the continuous model. Indeed, a careful Fourier analysis of those initial data shows that their discrete Fourier transform vanishes quadratically at the points $\xi = \pm\pi/2h$ and $\xi = \pm\pi/h$. The choice of the ratio $1/4$ between the two meshes plays a key role at this level.

2. Fourier analysis of slowly oscillating sequences

In this section we obtain explicit properties of the discrete Fourier transform of slowly oscillating sequences (SOS). The SOS on the fine grid $h\mathbb{Z}$ are those which are obtained from the coarse grid $4h\mathbb{Z}$ by an interpolation process. Obviously there is a one to one correspondence between the coarse grid sequences and the space $\mathbb{C}_4^{h\mathbb{Z}} = \{\psi \in \mathbb{C}^{h\mathbb{Z}}\colon \operatorname{supp}\psi \subset 4h\mathbb{Z}\}$. We introduce the extension operator E:

$$(E\psi)\big((4j+r)h\big) = \frac{4-r}{4}\psi(4jh) + \frac{r}{4}\psi\big((4j+4)h\big), \quad \forall j \in \mathbb{Z}, r = \overline{0,3},\ \psi \in \mathbb{C}_4^{h\mathbb{Z}}. \tag{3}$$

Let V_4^h be the space of slowly oscillating sequences $V_4^h = \{E\psi\colon\ \psi \in C_4^{h\mathbb{Z}}\}$. We also consider the projection operator $\Pi : \mathbb{C}^{h\mathbb{Z}} \to \mathbb{C}_4^{h\mathbb{Z}}$:

$$(\Pi\phi)\big((4j+r)h\big) = \phi\big((4j+r)h\big)\delta_{4r}, \quad \forall j \in \mathbb{Z},\ r = \overline{0,3},\ \phi \in \mathbb{C}^{h\mathbb{Z}}, \tag{4}$$

where δ is the Kronecker's symbol. We remark that $E : C_4^{h\mathbb{Z}} \to V_4^h$ and $\Pi : V_4^h \to \mathbb{C}_4^{h\mathbb{Z}}$ are bijective linear maps satisfying $\Pi E = I_{\mathbb{C}_4^{h\mathbb{Z}}}$, $E\Pi = I_{V_4^h}$, where I_X denotes the identity operator on X. We now define $\widetilde{\Pi} = E\Pi : \mathbb{C}^{h\mathbb{Z}} \to V_4^h$, which acts as a smoothing operator and associates to each sequence on the fine grid a slowly oscillating sequence. As we said above the restriction of this operator to V_4^h is the identity. Concerning the discrete Fourier transform of a SOS by means of explicit computations one can prove that:

Lemma 2.1. *Let $\phi \in l^2(h\mathbb{Z})$. Then*

$$\widehat{\widetilde{\Pi}\phi}(\xi) = 4\cos^2(\xi h)\cos^2(\xi h/2)\widehat{\Pi\phi}(\xi). \tag{5}$$

Remark 1. A simpler construction may be done interpolating $2h\mathbb{Z}$ sequences. We then get $\widehat{\widetilde{\Pi}\varphi}(\xi) = 2\cos^2(\xi h/2) \times \widehat{\Pi\varphi}(\xi)$. This cancels the spurious numerical solutions at the frequencies $\pm\pi/h$, but not at $\pm\pi/2h$. In this case, as

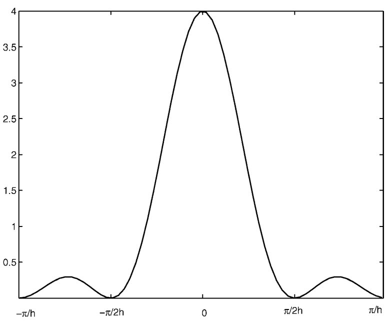

Fig. 1. The smoothing factor $4\cos^2(\xi h)\times\cos^2(\xi h/2)$.

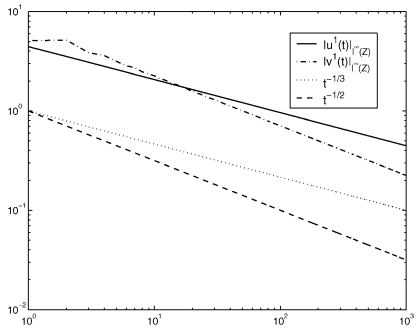

Fig. 2. Log–log plot of the time evolution of the l^∞ norms.

we proved in [3], the Strichartz estimates fail to be uniform on h. Thus we rather choose the ratio between grids to be $1/4$. The smoothing factor $4\cos^2(\xi h)\cos^2(\xi h/2)$ can be seen in Fig. 1.

Remark 2. The action the linear semigroup $\mathrm{e}^{\mathrm{i}t\Delta_h}$ on the subspace V_4^h can be seen in Fig. 2. For that let $h=1$ and δ_0 the sequence defined by $(\delta_0)_j=\delta_{0j}$. Choosing $u^1(0)=\delta_0$ and $v^1(0)=E\delta_0$ we observe the different behavior of the $l^\infty(\mathbb{Z})$ norm of $u^1(t)$ and $v^1(t)$. We point out that $\|u^1(t)\|_{l^\infty(\mathbb{Z})}$ and $\|v^1(t)\|_{l^\infty(\mathbb{Z})}$ behave like $t^{-1/3}$ respectively $t^{-1/2}$ as $t\to\infty$. In [7] the authors study the behavior of the linear Schrödinger equation on lattices obtaining the same long time results for $u^1(t)$.

3. Estimates of the linear semigroup

As we proved in [3], there is no gain (uniformly in h) of integrability of the linear semigroup $\mathrm{e}^{\mathrm{i}t\Delta_h}$. However, there are subspaces of $\mathbb{C}^{h\mathbb{Z}}$, namely V_4^h, where the linear semigroup has appropriate decay properties, uniformly on $h>0$. The main results we get are the following

Theorem 3.1. *Let $p\geqslant 2$. The following properties hold*:

(i) $\|\mathrm{e}^{\mathrm{i}t\Delta_h}\widetilde{\Pi}\varphi\|_{l^p(h\mathbb{Z})}\lesssim |t|^{-1/2(1/p'-1/p)}\|\widetilde{\Pi}\varphi\|_{l^{p'}(h\mathbb{Z})}$ *for all $\varphi\in l^{p'}(h\mathbb{Z})$, $h>0$ and $t\neq 0$.*

(ii) *For every sequence $\varphi\in l^2(h\mathbb{Z})$, the function $t\to \mathrm{e}^{\mathrm{i}t\Delta_h}\widetilde{\Pi}\varphi$ belongs to $L^q(\mathbb{R},l^r(h\mathbb{Z}))\cap C(\mathbb{R},l^2(h\mathbb{Z}))$ for every admissible pair (q,r). Furthermore $\|\mathrm{e}^{\mathrm{i}t\Delta_h}\widetilde{\Pi}\varphi\|_{L^q(\mathbb{R},l^r(h\mathbb{Z}))}\lesssim\|\widetilde{\Pi}\varphi\|_{l^2(h\mathbb{Z})}$ uniformly on $h>0$.*

(iii) *Let (q,r), $(\tilde q,\tilde r)$ be two admissible pairs. Then $\|\int_{s<t}\mathrm{e}^{\mathrm{i}(t-s)\Delta_h}\widetilde{\Pi}F(s)\,\mathrm{d}s\|_{L^q(\mathbb{R},l^r(h\mathbb{Z}))}\lesssim\|\widetilde{\Pi}F\|_{L^{\tilde q'}(\mathbb{R},l^{\tilde r'}(h\mathbb{Z}))}$ for all $F\in L^{\tilde q'}(\mathbb{R},l^{\tilde r'}(h\mathbb{Z}))$, uniformly in $h>0$.*

Sketch of the proof. Using that $\mathrm{e}^{\mathrm{i}t\Delta_h}=\mathrm{e}^{\mathrm{i}(t/h^2)\Delta_1}$, by scaling, we can assume that $h=1$. By (5) we obtain

$$\big(\mathrm{e}^{\mathrm{i}t\Delta_1}\widetilde{\Pi}\varphi\big)_j=\int_{-\pi}^{\pi}4\cos^2(\xi)\cos^2(\xi/2)\,\mathrm{e}^{-4\mathrm{i}t\sin^2(\xi/2)}\widehat{\Pi\varphi}(\xi)\mathrm{e}^{\mathrm{i}j\xi}\,\mathrm{d}\xi=\mathrm{def}\big(T(t)(\Pi\varphi)\big)_j.$$

It is sufficient to show that $T(t)$ maps $l^2(\mathbb{Z})$ to $l^2(\mathbb{Z})$ and $l^1(\mathbb{Z})$ to $l^\infty(\mathbb{Z})$ with appropriate norm decay in t. The case $p=2$ follows by Plancherel's identity. In the case $p=1$ we write $T(t)$ as a convolution operator $T(t)\psi=K^t*\psi$ where $\widehat{K^t}(\xi)=4\,\mathrm{e}^{-4\mathrm{i}t\sin^2\xi/2}\cos^2\xi\cos^2(\xi/2)$. It remains to prove that $\|K^t\|_{l^\infty(\mathbb{Z})}\lesssim 1/\sqrt{t}$. Using that $(4\sin^2(\xi/2))''=2\cos(\xi)$, by [5] (Corollary 2.9, p. 46) we obtain

$$\|K^t\|_{l^\infty(\mathbb{Z})}\lesssim\frac{1}{\sqrt{t}}\Big[\big\||\cos(\xi)|^{3/2}\cos^2(\xi/2)\big\|_{L^\infty([-\pi,\pi])}+\int_{-\pi}^{\pi}\big|\big(|\cos(\xi)|^{3/2}\cos^2(\xi/2)\big)'\big|\,\mathrm{d}\xi\Big]\lesssim\frac{1}{\sqrt{t}}.$$

Observe that the operators $T(t)$ satisfy $(T(t))^* = T(-t)$ for all real t. As a consequence we obtain $\|T(t)(T(s))^*\psi\|_{l^\infty(\mathbb{Z})} \lesssim |t-s|^{-1/2}\|\psi\|_{l^1(\mathbb{Z})}$, for all $t \neq s$ and $\psi \in l^1(\mathbb{Z})$. We are in the hypothesis of [4] (Theorem 1.2, p. 956). This implies that for all admissible pairs (q, r) and $(\tilde{q}, \tilde{r})$ we get

$$\left\|T(\cdot)f\right\|_{L^q(\mathbb{R},l^r(\mathbb{Z}))} \lesssim \|f\|_{l^2(\mathbb{Z})} \quad \text{and} \quad \left\|\int_{s<t} T(t-s)F(s)\,\mathrm{d}s\right\|_{L^q(\mathbb{R},l^r(\mathbb{Z}))} \lesssim \|F\|_{L^{\tilde{q}}(\mathbb{R},l^{\tilde{r}}(\mathbb{Z}))}. \tag{6}$$

Finally we use the definition of $T(t)$ in order to obtain the estimates for $\mathrm{e}^{\mathrm{i}t\Delta_1}$. □

Concerning the local smoothing properties we can prove that

Theorem 3.2. *Let $r \in (1, 2]$. Then, for all $f \in l^r(h\mathbb{Z})$, uniformly on $h > 0$, we have*

$$\sup_{j\in\mathbb{Z}} \int_{-\infty}^{\infty} \left|\left(D^{1-1/r}\mathrm{e}^{\mathrm{i}t\Delta_h}\widetilde{\Pi} f\right)_j\right|^2 \mathrm{d}t \lesssim \|\widetilde{\Pi} f\|^2_{l^r(h\mathbb{Z})}. \tag{7}$$

Sketch of the proof. By scaling we can assume that $h = 1$. Using that $\mathrm{e}^{\mathrm{i}t\Delta_1}\widetilde{\Pi} f = T(t)\Pi f$ it is sufficient to prove that $\sup_{j\in\mathbb{Z}} \int_{-\infty}^{\infty} |(D^{1-1/r}T(t)\psi)_j|^2\,\mathrm{d}t \lesssim \|\psi\|^2_{l^r(\mathbb{Z})}$. We introduce the continuous extension of T:

$$\left(T_1(t)\varphi\right)(x) = \int_{-\pi}^{\pi} \mathrm{e}^{-4\mathrm{i}t\sin^2\frac{\xi}{2}}\hat{\varphi}(\xi)\,\mathrm{e}^{\mathrm{i}x\xi}\cos^2\xi\cos^2(\xi/2)\,\mathrm{d}\xi. \tag{8}$$

It is sufficient to prove that

$$\sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} \left|\left(D^{1-1/r}T_1(t)\varphi\right)(x)\right|^2 \mathrm{d}t \lesssim \|\varphi\|^2_{L^r(\mathbb{R})} \tag{9}$$

for all $\varphi \in L^r(\mathbb{R})$ with $\operatorname{supp}\hat{\varphi} \in [-\pi, \pi]$. Using Sobolev's imbedding $L^r(\mathbb{R}) \hookrightarrow H^{1/2-1/r}(\mathbb{R})$ the inequality (9) may be reduced to the following one

$$\sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} \left|\left(T_1(t)\varphi\right)(x)\right|^2 \mathrm{d}t \lesssim \|D^{-1/2}\varphi\|^2_{L^2(\mathbb{R})} \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}). \tag{10}$$

Applying the results of [5] (Theorem 4.1, p. 54) to T_1 we get

$$\sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} \left|\left(T_1(t)\varphi\right)(x)\right|^2 \mathrm{d}t \lesssim \int_{-\pi}^{\pi} \frac{|\hat{f}(\xi)|^2\cos^4\xi\cos^4(\xi/2)}{|\sin\xi|}\,\mathrm{d}\xi \lesssim \int_{-\pi}^{\pi} \frac{|\hat{f}(\xi)|^2}{|\xi|}\,\mathrm{d}\xi \lesssim \|D^{-1/2}f\|^2_{L^2(\mathbb{R})}. \quad \square \tag{11}$$

4. A conservative approximation of the NSE

We concentrate on the semilinear NSE equation in $\mathbb{R}$ with repulsive power law nonlinearity:

$$\mathrm{i}u_t + \Delta u = |u|^p u, \quad x \in \mathbb{R},\ t \in \mathbb{R}; \qquad u(0, x) = \varphi(x), \quad x \in \mathbb{R}. \tag{12}$$

As proved in [8], (12) is globally well posed for all $\varphi \in L^2(\mathbb{R})$ and $p \in [0, 4)$. Consider the semi-discretization

$$\mathrm{i}u_t^h + \Delta_h u^h = \widetilde{\Pi} f(u^h), \quad t \in \mathbb{R}; \qquad u^h(0) = \widetilde{\Pi}\varphi^h, \tag{13}$$

where $f(u^h)$ is a suitable approximation of $|u|^p u$ with $0 < p < 4$. In order to prove the global well-posedness of (13), we need to guarantee the conservation of the $l^2(h\mathbb{Z})$ norm of solutions, a property that the solutions of NSE satisfy. For that the nonlinear term $f(u^h)$ has to be chosen such that $(\widetilde{\Pi} f(u^h), u^h)_{l^2(h\mathbb{Z})} \in \mathbb{R}$. These property is guaranteed with the choice

$$\left(f(u^h)\right)_{4j} = g\left(\left(u^h_{4j} + \sum_{r=1}^{3} \tfrac{4-r}{4}(u^h_{4j+r} + u^h_{4j-r})\right)/4\right); \qquad g(s) = |s|^p s. \tag{14}$$

The following holds:

Theorem 4.1. *Let $p \in (0, 4)$, $q = 4(p + 2)/p$ and $f : \mathbb{C}^{h\mathbb{Z}} \to \mathbb{C}^{h\mathbb{Z}}$ be as above. Then for every $\varphi^h \in l^2(h\mathbb{Z})$, there exists a unique global solution $u^h \in C(\mathbb{R}, l^2(h\mathbb{Z})) \cap L^q_{\mathrm{loc}}(\mathbb{R}; l^{p+2}(h\mathbb{Z}))$ of* (13) *which satisfies for all finite interval I, uniformly on $h > 0$, the following estimates*

$$\|u^h\|_{L^\infty(\mathbb{R},l^2(h\mathbb{Z}))} \leqslant \|\widetilde{\Pi}\varphi\|_{l^2(h\mathbb{Z})} \quad \text{and} \quad \|u^h\|_{L^q(I,l^{p+2}(h\mathbb{Z}))} \leqslant c(I)\|\widetilde{\Pi}\varphi\|_{l^2(h\mathbb{Z})}. \tag{15}$$

Sketch of the proof. Local existence and uniqueness are consequence of the Strichartz estimates (Theorem 3.1) and a fixed point argument. The fact that $(\widetilde{\Pi} f(u^h), u^h)_{l^2(h\mathbb{Z})}$ is real guarantees the conservation of the discrete energy $h\sum_{j\in\mathbb{Z}}|u_j(t)|^2$. This allows excluding finite-time blow-up. □

In the sequel we consider the piecewise constant interpolator I_h. We choose $(\varphi_j^h)_{j\in\mathbb{Z}}$, an approximation of the initial data $\varphi\in L^2(\mathbb{R})$, such that $I_h\widetilde{\Pi}\varphi^h \rightharpoonup \varphi$ weakly in $L^2(\mathbb{R})$.

The main convergence result is the following

Theorem 4.2. *Let u^h be the unique solution of* (13). *Then the sequence $I_h u^h$ satisfies*

$$I_h u^h \overset{\star}{\rightharpoonup} u \quad \text{in } L^\infty\big(\mathbb{R}, L^2(\mathbb{R})\big), \qquad I_h u^h \rightharpoonup u \quad \text{in } L^q_{\mathrm{loc}}\big(\mathbb{R}, L^{p+2}(\mathbb{R})\big), \tag{16}$$

$$I_h u^h \to u \quad \text{in } L^2_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}), \qquad I_h\widetilde{\Pi} f(u^h) \rightharpoonup |u|^p u \quad \text{in } L^{q'}_{\mathrm{loc}}\big(\mathbb{R}, L^{(p+2)'}(\mathbb{R})\big) \tag{17}$$

where u is the unique solution of NSE and $2/q = 1/2 - 1/(p+2)$.

Sketch of the proof. Using the result of Theorem 3.2 with $r=2$ for the initial data and $r=(p+2)'$ for the nonlinearity we first prove that $\|I_h u^h\|_{L^2_{\mathrm{loc}}(\mathbb{R}, H^{1/(p+2)}_{\mathrm{loc}}(\mathbb{R}))} \leqslant \|\widetilde{\Pi}\varphi^h\|_{l^2_h(\mathbb{Z})}$. Then, by a compactness argument we can extract a subsequence converging locally strongly in $L^1_{x,t}$. This, together with the uniform (with respect to h) estimates of Theorem 4.1, suffices to obtain the stated convergence results. In particular, it suffices to pass to the limit in the nonlinear term and to identify the limit as the solution of NSE. □

Remark 3. Our method works similarly in the critical case $p=4$ for small initial data.

Remark 4. The techniques and results of this paper extend to several space dimensions.

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