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Partial Differential Equations

Nonlinear Schrödinger equations: concentration on weighted geodesics in the semi-classical limit

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Abstract

We consider the problem

$$\varepsilon^2\Delta u - V(x)u + u^p = 0, \quad u > 0, \ u \in H^1(\mathbb{R}^2),$$

where $p > 1$, $\varepsilon > 0$ is a small parameter and V is a uniformly positive, smooth potential. Let Γ be a closed curve, nondegenerate geodesic relative to the weighted arclength $\int_\Gamma V^\sigma$, where $\sigma = \frac{p+1}{p-1} - \frac{1}{2}$. We prove the existence of a solution u_ϵ concentrating along the whole of Γ, exponentially small in ε at any positive distance from it, provided that ε is small and away from certain critical numbers. This proves a conjecture raised in [A. Ambrosetti, A. Malchiodi, W.-M. Ni, Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres, Part I, Commun. Math. Phys. 235 (2003) 427–466] in the two-dimensional case. ***To cite this article: M. del Pino et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Equations de Schrödinger non linéaires : Concentration sur des géodésiques pondérées dans la limite semi-classique. On considère le problème

$$\varepsilon^2\Delta u - V(x)u + u^p = 0, \quad u > 0, \ u \in H^1(\mathbb{R}^2),$$

avec $p > 1$, où $\varepsilon > 0$ est un petit paramètre et V est un potentiel régulier, uniformément positif. Soit Γ une courbe fermée formant une géodésique non dégénérée relativement à la longueur pondérée $\int_\Gamma V^\sigma$, avec $\sigma = \frac{p+1}{p-1} - \frac{1}{2}$. Nous démontrons l'existence d'une solution u_ϵ qui se concentre le long de la courbe Γ tout entière, exponentiellement petite en ε à toute distance positive de Γ, pourvu que ε soit petit et évite certaines valeurs critiques. Ceci répond affirmativement à une conjecture énoncée

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dans [A. Ambrosetti, A. Malchiodi, W.-M. Ni, Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres, Part I, Commun. Math. Phys. 235 (2003) 427–466] dans le cas bi-dimensionnel. ***Pour citer cet article : M. del Pino et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Version française abrégée

Soit $V(x)$ une fonction régulière avec $\inf_{x\in\mathbb{R}^2}, V(x)>0$. On considère le problème

$$\varepsilon^2\Delta u - V(x)u + u^p = 0, \quad u>0,\ u\in H^1(\mathbb{R}^N), \tag{1}$$

correspondant aux ondes stationnaires de l'équation de Schrödinger non linéaire standard. Une attention considérable a été apportée au cours des dernières années au problème de la construction de telles ondes stationnaires dans la limite dite *semi-classique* $\varepsilon\to 0$, à la suite du travail précurseur de Floer et Weinstein [9] en une dimension d'espace. Il s'agit essentiellement de trouver des solutions qui se concentrent en un ou plusieurs de l'espace quand $\varepsilon\to 0$. Une question importante consiste à déterminer si des solutions présentant des propriétés de concentration sur des ensemble de dimension plus élevée existent.

Soit Γ une courbe fermée régulière dans $\mathbb{R}^2$, de longueur totale ℓ, et (t,θ) ses coordonnées naturelles normales et curvilignes décrivant tous les points à proximité de la courbe. Soit w l'unique solution de

$$w'' - w + w^p = 0, \quad w>0, \quad w'(0)=0, \quad w(\pm\infty)=0. \tag{2}$$

Nous supposons de plus que Γ est *stationnaire et non-dégénérée* pour la fonctionnelle d'aire pondérée $\int_\Gamma V^\sigma$, $\sigma = \frac{p+1}{p-1}-\frac{1}{2}$. En d'autres termes, Γ est une géodésique non dégénérée pour la métrique conforme $V^\sigma\,\mathrm{d}x^2$. Notre principal résultat est le suivant.

Théorème 0.1. *Il existe une constante $\lambda_*>0$ telle que, étant donné $c>0$, il existe $\varepsilon_0>0$ pour lequel, pour tout $0<\varepsilon<\varepsilon_0$ vérifiant la condition de gap*

$$|\varepsilon^2k^2-\lambda_*|\geqslant c\varepsilon, \tag{3}$$

le problème (6) *a une solution strictement positive u_ϵ qui, près de Γ, prend la forme*

$$u_\epsilon(t,\theta) = V(0,\theta)^{1/(p-1)} w\left(V(0,\theta)^{1/2}\frac{t}{\varepsilon}\right)\bigl(1+\mathrm{o}(1)\bigr), \tag{4}$$

et est d'ordre $\mathrm{O}(\mathrm{e}^{-\delta/(2\varepsilon)})$ *en dehors de tout δ-voisinage de Γ.*

Ce résultat démontre la conjecture soulevée dans [2] dans le cas bi-dimensionnel. La preuve est basée sur une réduction de Lyapunov–Schmidt infinie : par un changement de variables qui absorbe ε par dilatation dans l'expression de l'opérateur de Laplace, c'est-à-dire en remplaçant $u(y)$ par $u(\varepsilon y)$, l'équation devient

$$\Delta u + V(\varepsilon y)u + u^p = 0.$$

Soit $(s,z)=\varepsilon^{-1}(t,\theta)$ les coordonnées naturelles dilatées. Pour $\alpha(\theta)=V(0,\theta)^{1/(p-1)}$, $\beta(\theta)=V(0,\theta)^{1/2}$, nous écrivons

$$u(s,z) = \alpha(\varepsilon z)v\bigl(\beta(\varepsilon z)\bigl(x-f(\varepsilon z)\bigr),z\bigr)$$

où f est une fonction ℓ-périodique donnée. Dans de telles coordonnées, nous recherchons la solution sous la forme

$$v(x,z) = w(x) + \varepsilon e(\varepsilon z)Z(x) + \phi(x,z)$$

près de la courbe Γ, où e est une deuxième fonction donnée, et ϕ est $L^2(\mathrm{d}x)$-orthogonale à $w_x(x)$ et $Z(x)$ pour tout z. Ici Z est la première fonction propre dans $L^2(\mathbb{R})$ du problème $Z'' + pw^{p-1}Z - Z = \lambda Z$. Une version projetée du problème non-linéaire résultant pour ϕ est tout d'abord résolue par une méthode d'application contractante basée sur l'invertibilitée uniforme des opérateurs linéaires mis en jeu. Après cette procédure, le problème complet est réduit à un système non-linéaire, non local, du second ordre, d'EDOs pour la paire (f, e) qui peut être résolu, grâce à notre hypothèse sur la courbe, par le théorème du point fixe de Schauder.

1. Introduction and statement of the main result

We consider *standing waves* for a nonlinear Schrödinger equation in $\mathbb{R}^N$ of the form

$$-\mathrm{i}\varepsilon\frac{\partial\psi}{\partial t} = \varepsilon^2\Delta\psi - Q(x)\psi + |\psi|^{p-1}\psi \tag{5}$$

where $p > 1$, namely solutions of the form $\psi(t,x) = \exp(\mathrm{i}\lambda\varepsilon^{-1}t)u(x)$. Assuming that the amplitude $u(x)$ is positive and vanishes at infinity, we see that this ψ satisfies (5) if and only if u solves the nonlinear elliptic problem

$$\varepsilon^2\Delta u - V(x)u + u^p = 0, \quad u > 0,\ u \in H^1(\mathbb{R}^N), \tag{6}$$

where $V(x) = Q(x) + \lambda$. We assume that V is smooth with $\inf_{x\in\mathbb{R}^2} V(x) > 0$. Considerable attention has been paid in recent years to the problem of construction of standing waves in the so-called *semi-classical limit* ($\varepsilon \to 0$). In the pioneering work [9], Floer and Weinstein constructed positive solutions to this problem when $p = 3$ and $N = 1$ with concentration taking place near a given point x_0 with $V'(x_0) = 0$, $V''(x_0) \neq 0$, being exponentially small in ε outside any neighborhood of x_0. This result has been subsequently extended to higher dimensions to the construction of solutions exhibiting high concentration around one or more points of space under various assumptions on the potential and the nonlinearity by many authors. We refer the reader for instance to [1,5–7, 10–12,17,18].

An important question is whether solutions exhibiting concentration on higher dimensional sets exist. In a related Neumann problem with concentration on the boundary this issue has been analyzed in [14–16]. The radial case has been considered in the works [2–4]. In [2], Ambrosetti, Malchiodi and Ni have considered the case of $V = V(|x|)$, also treated in [3,4], and constructed radial solutions $u_\varepsilon(|x|)$ exhibiting concentration on a sphere $|x| = r_0$ under the assumption that $r_0 > 0$ is a non-degenerate critical point of $M(r) = r^{N-1}V^\sigma(r)$ where $\sigma = \frac{p+1}{p-1} - \frac{1}{2}$. The asymptotic profile of this solution is $u_\varepsilon(r) \sim V(r_0)^{1/(p-1)}w(V(r_0)^{1/2}\varepsilon^{-1}(r - r_0))$, where w is the unique solution of

$$w'' - w + w^p = 0, \quad w > 0, \quad w'(0) = 0, \quad w(\pm\infty) = 0. \tag{7}$$

In [2] it is conjectured that this type of phenomenon takes place (at least along a sequence $\varepsilon = \varepsilon_n \to 0$) whenever the sphere $|x| = r_0$ is replaced by a closed hypersurface Γ, which is *stationary and non-degenerate* for the weighted area functional $\int_\Gamma V^\sigma$. In this note we prove the validity of this conjecture in dimension $N = 2$.

Let Γ be a closed smooth curve in $\mathbb{R}^2$ and $\ell = |\Gamma|$ its total length. We consider the natural parametrization $\gamma(\theta)$ of Γ with positive orientation, where θ denotes the arclength parameter, $\nu(\theta)$ the outer unit normal. Points y near Γ can then be represented in the form

$$y = \gamma(\theta) + t\nu(\theta). \tag{8}$$

We say that Γ is *stationary and non-degenerate* for the weighted length $J(\Gamma) = \int_\Gamma V^\sigma$ if first variation vanishes on Γ and second variation is non-singular. If k denotes curvature of Γ and we denote $V(t,\theta) = V(y)$, y given by (8), these assumptions are respectively equivalent to the pointwise relation

$$\sigma V_t(0,\theta) = -k(\theta)V(0,\theta) \quad \text{for all } \theta \in (0,\ell), \tag{9}$$

and to the fact that the only ℓ-periodic solution of the differential equation

$$(V^\sigma h')' - \big[(V^\sigma)_{tt} - 2V^\sigma k^2\big]h = 0 \tag{10}$$

is $h \equiv 0$. Our main result is the following.

Theorem 1.1. *Let Γ be a nondegenerate, stationary curve for the weighted length functional $\int_\Gamma V^\sigma$, as described above. Then there is a constant $\lambda_* > 0$ such that the following holds. Given $c > 0$ there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ satisfying the condition*

$$|\varepsilon^2 k^2 - \lambda_*| \geqslant c\varepsilon, \tag{11}$$

problem (6) *has a positive solution u_ϵ which near Γ has the form*

$$u_\epsilon(y) = V(0,\theta)^{1/(p-1)} w\left(V(0,\theta)^{1/2}\frac{t}{\varepsilon}\right)\big(1 + \mathrm{o}(1)\big), \tag{12}$$

and is exponentially small in ε outside any neighborhood of Γ.

2. Sketch of proof of Theorem 1.1

We sketch here the proof of the theorem, whose details are provided in [8]. The first step consists of building an approximation to a solution of the problem which involves as parameters some functions to be later adjusted. Expanding the local variables (t,θ) defined on a neighborhood of Γ to $(s,z) = \varepsilon^{-1}(t,\theta)$, we see that the equation becomes in terms of these variables

$$u_{zz} + u_{ss} + B_1(u) - V(\varepsilon s, \varepsilon z)u + u^p = 0 \tag{13}$$

for, say $|s| \leqslant \frac{\delta}{\varepsilon}$, where u is $\ell\varepsilon^{-1}$-periodic in the z variable and

$$B_1(u) = u_{zz}\left[1 - \frac{1}{(1+\varepsilon k(\varepsilon z)s)^2}\right] + \frac{\varepsilon k(\varepsilon z)u_s}{1+\varepsilon k(\varepsilon z)s} + \frac{\varepsilon^2 s k'(\varepsilon z)u_z}{(1+\varepsilon k(\varepsilon z)s)^3}.$$

We make a further change of variables: given an ℓ-periodic function $f(\theta)$ we set

$$u(s,z) = \alpha(\varepsilon z)v\big(\beta(\varepsilon z)\big(x - f(\varepsilon z)\big), z\big)$$

where

$$\alpha(\theta) = V(0,\theta)^{1/(p-1)}, \qquad \beta(\theta) = V(0,\theta)^{1/2}.$$

The equation written in terms of v becomes now

$$S(v) \equiv \beta^{-2}v_{zz} + v_{xx} + B_2(v) + v^p - v = 0 \tag{14}$$

where $B_2(v)$ is a linear differential operator inherited from B_1 for derivatives written in terms of the new variables, and of the difference $V(\varepsilon s, \varepsilon z) - V(0, \varepsilon z)$, so that all terms in this operator carry at least one power of ε in front. In these coordinates, we take $w(x)$ as our first approximation for a solution, which yields then an error $E_0 = S(w)$ of size ε. Examining closely this term, we see that its part of order ε turns out to be precisely

$$E_1 = \varepsilon\beta^{-1}\big[kw_x - \beta^{-2}V_t(0,\varepsilon z)xw\big] - \varepsilon\beta^{-2}V_t(0,\varepsilon z)fw.$$

Our assumption (9) turns out to be equivalent to this E_1 being $L^2(\mathrm{d}x)$ orthogonal to $w_x(x)$ for each z. This allows to improve the approximation to order ε^2 by adding to w the unique solution $\phi = \varepsilon w_1$ orthogonal to w_x of the problem

$$\phi_{xx} - \phi + pw^{p-1}\phi = E_1.$$

To make the full first approximation we need a further parameter. We consider now a new periodic function $e(\theta)$ and consider the approximation near the curve to be

$$\mathtt{w}(x,z) = w(x) + \varepsilon w_1(x,z) + \varepsilon e(\varepsilon z) Z(x).$$

Here Z is the first eigenfunction in $L^2(\mathbb{R})$ of the problem $Z'' + pw^{p-1}Z - Z = \lambda_0 Z$. We search a solution of (13) of the form $v = \mathtt{w} + \phi$. We assume that f and e are ℓ-periodic functions in $H^2(0,\ell)$ which satisfy in $(0,\ell)$ the bounds

$$\|f''\|_2 + \|f'\|_\infty + \|f\|_\infty \leqslant \varepsilon^{1/2}, \qquad \varepsilon^2\|e''\|_2 + \varepsilon\|e'\|_\infty + \|e\|_\infty \leqslant \varepsilon^{1/2}. \tag{15}$$

The expressions involved only make sense in principle for $|x| < 2\delta\varepsilon^{-1}$ for some $\delta > 0$, however we can idealize both $\mathtt{w}$ and ϕ as defined in the whole strip $\mathcal{S} = \mathbb{R} \times [0, \ell\varepsilon^{-1}]$. Consider a smooth cut-off function $\chi(|x|\varepsilon^{-1})$ with $\chi(t) = 1$, $t < \delta$, $x(t) = 0$, $t > 2\delta$, and the problem for ϕ, which reduces to (13) for $|x| < \delta\varepsilon^{-1}$,

$$L(\phi) = \chi E + N(\phi), \quad \phi \in H^2(\mathcal{S}) \tag{16}$$

with periodic boundary conditions on ∂S. Here

$$L(\phi) = \beta^{-2}\phi_{zz} + \phi_{xx} - \phi + pw^{p-1}\phi + \chi B_2(\phi), \tag{17}$$

$$N(\phi) = \chi\big[(\mathtt{w}+\phi)^p - \mathtt{w}^p - p\mathtt{w}^{p-1}\phi\big]. \tag{18}$$

This problem is of course not equivalent in the entire space to the original one, but a *gluing reduction* procedure makes it *essentially so*, to the expense of perturbing N by a an exponentially small (in ε) nonlocal operator in ϕ which can be neglected for the whole remaining of the argument.

We consider first the following projected problem in $H^2(S)$: given $e, f \in H^2(0,\ell)$, find functions $\phi \in H^2(S)$, $c, d \in L^2(0,\ell)$ such that

$$L(\phi) = \chi E + N(\phi) + c(\varepsilon z)\chi w_x + e(\varepsilon z)\chi Z, \quad \phi \in H^2(\mathcal{S}), \tag{19}$$

$$\phi(x,0) = \phi\left(x, \frac{\ell}{\varepsilon}\right), \quad \phi_z(x,0) = \phi_z\left(x, \frac{\ell}{\varepsilon}\right) \quad \text{for all } -\infty < x < +\infty, \tag{20}$$

$$\int_{-\infty}^{\infty} \phi(x,z) w_x(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \phi(x,z) Z(x)\,\mathrm{d}x = 0 \quad \text{for all } 0 < z < \frac{\ell}{\varepsilon}. \tag{21}$$

Here $E = S(\mathtt{w})$. The orthogonality condition makes the operator L to have a uniformly bounded inverse between $L^2(S)$ and $H^2(S)$-periodic in z. This, analyzing the size and form of the errors created, and an application of contraction mapping principle yield the existence of a number $D > 0$, such that for all sufficiently small ε and any (f,e) satisfying (15), problem (19), (20) has a unique solution $\phi = \phi(f,e)$ which satisfies $\|\phi\|_{H^2(S)} \leqslant D\varepsilon^{3/2}$. The last part of the proof is to set up equations for f and d which are equivalent to making the coefficients c, d in (19) identically zero. This is achieved by simply multiplying the equations respectively by w_x and Z, and integrating in x. After an integration by parts, using the orthogonality conditions satisfied by ϕ, we find that $(c,d) \equiv (0,0)$ is equivalent to a system for (f,e) of the form

$$\mathcal{L}_1(f) = f'' + \gamma_1 f' + \gamma_2 f = \gamma_3 e + \varepsilon M_{1\varepsilon}(f,e), \tag{22}$$

$$\mathcal{L}_2(e) = \varepsilon^2\big[1 + \varepsilon f\gamma_4 + \mathrm{O}(\varepsilon)\big]\beta^{-2}e'' + \varepsilon^2\gamma_5 e' + \lambda_0 e = \varepsilon\gamma_6 e^2 + \varepsilon\gamma_7 fe + \varepsilon\gamma_8 + \varepsilon^2 M_{2\varepsilon}(f,e). \tag{23}$$

Here $\gamma_i(\theta)$ are smooth, ℓ-periodic functions. The operators $M_{1\varepsilon}$ and $M_{1\varepsilon}$ are uniformly bounded in $L^2(0,\ell)$, on the region of (f,e) satisfying constraints (15). Moreover, for each fixed ε they define compact operators. The coefficients γ_i can be computed explicitly: the operator $\mathcal{L}_1(f)$ corresponds precisely to $V^{-\sigma}$ times that in the left-hand side of Eq. (10), and it is therefore invertible with periodic boundary conditions, with bounded inverse between $L^2(0,\ell)$ and $H^2(0,\ell)$. The operator $\mathcal{L}_2(e)$ is also invertible with periodic boundary conditions: 0 is $c\varepsilon$-away from its spectrum thanks to the condition (11) if the choice $\lambda_* = \lambda_0 \frac{1}{4\pi^2}(\int_0^\ell V(0,\theta)^{1/2}\,\mathrm{d}\theta)^2$ is made, and one uses classical asymptotic formulas for eigenvalues of periodic Sturm–Liouville operators found for instance in [13]. More precisely, we find the validity of the following estimates for the inverse $e = \mathcal{L}_2^{-1}(d)$.

$$\varepsilon^2\|e''\|_2 + \varepsilon\|e'\|_2 + \|e\|_2 \leqslant C\varepsilon^{-1}\|d\|_2, \qquad \|e\|_\infty \leqslant C\big[\|d\|_2 + \|d'\|_2\big].$$

Using these estimates, it is not hard to construct an invariant region inside constraints (15) for the fixed point operator obtained from (22)–(23) after applying inverses of $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. By applying Schauder's fixed point theorem, one completes the proof. □

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References

[1] A. Ambrosetti, M. Badiale, S. Cingolani, Semiclassical states of nonlinear Schrödinger equations, Arch. Rational Mech. Anal. 140 (1997) 285–300.
[2] A. Ambrosetti, A. Malchiodi, W.-M. Ni, Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres, Part I, Comm. Math. Phys. 235 (2003) 427–466.
[3] M. Badiale, T. D'Aprile, Concentration around a sphere for a singularly perturbed Schrödinger equation, Nonlinear Anal. 49 (2002) 947–985.
[4] V. Benci, T. D'Aprile, The semiclassical limit of the nonlinear Schrödinger equation in a radial potential, J. Differential Equations 184 (2002) 109–138.
[5] J. Byeon, Z.-Q. Wang, Standing waves with a critical frequency for nonlinear Schrödinger equations, Arch. Rational Mech. Anal. 65 (2002) 295–316.
[6] M. del Pino, P. Felmer, Semi-classical states for nonlinear Schrödinger equations, J. Funct. Anal. 149 (1997) 245–265.
[7] M. del Pino, P. Felmer, Semi-classical states of nonlinear Schrödinger equations: a variational reduction method, Math. Ann. 324 (1) (2002) 1–32.
[8] M. del Pino, M. Kowalczyk, J. Wei, Concentration on curves for nonlinear Schrödinger equations, Comm. Pure Appl. Math., in press.
[9] A. Floer, A. Weinstein, Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential, J. Funct. Anal. 69 (1986) 397–408.
[10] M. Grossi, On the number of single-peak solutions of the nonlinear Schrödinger equation, Ann. Inst. H. Poincaré Anal. Non Lineaire 19 (2002) 261–280.
[11] L. Jeanjean, K. Tanaka, Singularly perturbed elliptic problems with superlinear or asymptotically linear nonlinearities, Calc. Var. Partial Differential Equations 21 (3) (2004) 287–318.
[12] X. Kang, J. Wei, On interacting bumps of semi-classical states of nonlinear Schrödinger equations, Adv. Differential Equations 5 (7–9) (2000) 899–928.
[13] B.M. Levitan, I.S. Sargsjan, Sturm–Liouville and Dirac operators, Math. Appl. (Soviet Ser.), vol. 59, Kluwer Academic, Dordrecht, 1991.
[14] A. Malchiodi, M. Montenegro, Boundary concentration phenomena for a singularly perturbed elliptic problem, Commun. Pure Appl. Math. 55 (2002) 1507–1568.
[15] A. Malchiodi, M. Montenegro, Multidimensional boundary layers for a singularly perturbed Neumann problem, Duke Math. J. 124 (1) (2004) 105–143.
[16] A. Malchiodi, Solutions concentrating at curves for some singularly perturbed elliptic problems, C. R. Math. Acad. Sci. Paris 338 (10) (2004) 775–780.
[17] Y.-G. Oh, On positive multibump states of nonlinear Schrödinger equation under multiple well potentials, Commun. Math. Phys. 131 (1990) 223–253.
[18] X. Wang, On concentration of positive bound states of nonlinear Schrödinger equations, Commun. Math. Phys. 153 (1993) 229–243.