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Differential Geometry

On linearization of planar three-webs and Blaschke's conjecture

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Abstract

We find relative differential invariants of orders eight and nine for a planar nonparallelizable 3-web such that their vanishing is necessary and sufficient for a 3-web to be linearizable. This solves the Blaschke conjecture for 3-webs. ***To cite this article: V.V. Goldberg, V.V. Lychagin, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Sur la linéarisation de 3-tissus plans et la conjecture de Blaschke. Nous présentons des invariants relatifs différentiels d'ordre huit et neuf pour un 3-tissu plan non parallélisable dont l'annulation est nécessaire et suffisante pour que la 3-tissu soit linéarisable. Ceci apporte une solution a la conjecture de Blaschke pour le problème de linéarisation des 3-tissus. ***Pour citer cet article : V.V. Goldberg, V.V. Lychagin, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Version française abrégée

Soit W_d un d-tissu donné par d feuilletages de courbes sur une variété bi-dimensionelle M^2. Le tissu W_d est linéarisable (rectifiable) si il est équivalent à un d-tissu linéaire, i.e., un d-tissu formé par d feuilletages de lignes droites sur un plan projectif.

Le problème d'établir un critère de linéarisabilité des tissus a été posé par Blaschke dans les années 1920 (voir par exemple son livre [2], §17). Il y conjecture que les conditions de linéarisabilité pour un 3-tissu W_3 doivent consister en quatre relations pour les invariants d'ordre neuf du tissu (pour des EDP d'ordre neuf).

Nous prouvons le résultat suivant :

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Théorème 0.1.

- *Un* 3*-tissu plan est linéarisable si et seulement si le système d'équations différentielles* (1) *a une solution lisse* $(\lambda_1, \lambda_2, \mu)$.
- *Les conditions de compatibilité pour le système* (1) *consistent en deux équations différentielles non-linéaires* (2) *du second ordre en la fonction* μ *seulement.*
- *Pour que le système d'équations différentielles* (2) *admette une solution, des conditions de compatibilité* (4) *doivent être ajoutées à ce système. Les conditions de compatibilité pour le système résultant sont équivalentes à l'existence de solutions réelles lisses pour le système d'équations algébriques* (5)*.*
- *En substituant une solution réelle lisse de* (5) *dans* (1)*, on obtient un système totalement intégrable pour les fonctions* (λ_1, λ_2).

Ce théorème implique qu'il y a 1040 invariants d'ordre inférieur ou égal à neuf (18 d'entre eux sont d'ordre huit) dont l'annulation est nécessaire et suffisante pour la linéarisabilité d'un 3-tissu W_3. Ceci prouve que l'estimation de Blaschke de la « codimension fonctionnelle » des orbites du 3-tissu linéarisable était juste, mais que le nombre d'invariants ne l'était pas. De plus, ce problème fait intervenir des invariants d'ordre huit qui ne correspondent pas à sa prédiction.

Nous établissons aussi une procédure pour tester la linéarisabilité d'un 3-tissu W_3 donné par une fonction de tissu $z = f(x, y)$ et nous appliquons cette procédure à un 3-tissu linéaire arbitraire.

1. Introduction

Let W_d be a d-web given by d foliations of curves on a two-dimensional manifold M^2. The web W_d is linearizable (rectifiable) if it is equivalent to a linear d-web, i.e., a d-web formed by d foliations of straight lines on a projective plane.

The problem of finding a criterion of linearizability of webs was posed by Blaschke in the 1920s (see, for example, his book [2], §17 and §42) who claimed that it is hopeless to find such a criterion. He made the conjectures that conditions of linearizability for a 3-web W_3 should consist of four relations for the ninth-order web invariants (four PDEs of ninth order) and those for a 4-web W_4 should consist of two relations for the fourth-order web invariants (two PDEs of fourth order).

In [1] the authors proved that the Blaschke conjecture on linearizability conditions for 4-webs was correct. In [1] a complete solution of the linearizability problem for d-webs, $d \geqslant 5$, was also presented. In [4] the linearizability conditions found in [1] were applied to check whether some known classes of 4-webs are linearizable.

In the present Note we continue to use the Akivis approach (see [1]) for establishing criteria of linearizability of 3-webs. In this approach, the linearizability problem is reduced to the solvability of a system of nonlinear PDEs on the components of the affine deformation tensor. This is the system of four nonlinear first-order PDEs with three unknown functions defined on the plane. In [3] the first obstruction for integrability of the system was found. In this Note we use results of [7] to investigate the integrability of the system and show that the obstruction found in [3] coincides with the Mayer bracket defined in [7].

We show that for nonparallelizable 3-webs, the solvability of the system indicated above is equivalent to the existence of real and smooth solutions of a system of five algebraic equations of degrees not exceeding 17, 18, 18, 24 and 24. This allows us:

(i) To find relative differential invariants whose vanishing leads to the linearizability of a 3-web W_3. This solves the *Blaschke problem* mentioned earlier on finding linearizability conditions in the form of invariants whose vanishing is necessary and sufficient for linearizability of a 3-web W_3. There are 1040 invariants of order not exceeding nine, and 18 of them have order eight. Note that the number of invariants can be different but there are always invariants of order eight. Note also that the Blaschke estimation of the 'functional codimension' of the orbits of the linearizable 3-webs was correct, but the number of invariants was not. Moreover, the problem has invariants of order eight, which do not match his prediction.

(ii) To establish an algorithm for determining whether a given 3-web W_3 is linearizable. This algorithm is based on investigation of the existence of a real solution of the five algebraic equations mentioned above.

The current Note contains the statements of our main results on linearization of planar 3-webs; see [5] for the proofs.

Note that the authors of [6] were trying to solve the linearizability problem for planar 3-webs. However, the results of [6] are incomplete because they do not contain all conditions. Moreover, the main (and only) example of a linearizable (in their approach) 3-web (see [6], Example (1), p. 2653) is not linearizable at all (see [5], Example 7, pp. 38–39).

2. Basic constructions

Let M^2 be a two-dimensional manifold, and suppose that a 3-web W_3 is given on M^2 by three differential 1-forms ω_1, ω_2, and ω_3 such that any two of them are linearly independent (see [3]). The forms ω_1, ω_2, and ω_3 can be normalized in such a way that the normalization condition $\omega_1 + \omega_2 + \omega_3 = 0$ holds. Because M^2 is a two-dimensional manifold, there is a unique differential 1-form γ such that $\mathrm{d}\omega_1 = \omega_1 \wedge \gamma$, $\mathrm{d}\omega_2 = \omega_2 \wedge \gamma$.

We call γ the *connection form*. The form γ determines the so-called Chern connection on M^2. The differential 2-form $\mathrm{d}\gamma$ does not depend on the web representation and is the curvature of the Chern connection. Let $\mathrm{d}\gamma = K\omega_1 \wedge \omega_2$. The function K is called the *web curvature*. This is a relative web-invariant of weight two.

It is easy to see that the foliations $\{\omega_1 = 0\}$, $\{\omega_2 = 0\}$, and $\{\omega_3 = 0\}$ are geodesic with respect to the Chern connection.

The problem of linearization of webs can be formulated as follows: *find a torsion-free flat connection such that the foliations of the web are geodesic with respect to this connection*.

Let γ be the Chern connection and let ∇ be a torsion-free connection such that the foliations $\{\omega_p = 0\}$, $p = 1, 2, 3$, are geodesic with respect to ∇. Then for the covariant differentials one has $d_\nabla = d_\gamma - T : \Omega^1(M) \to \Omega^1(M) \otimes \Omega^1(M)$, where the deformation tensor T is of the form

$$T = \left(T^1_{11}\omega_1 \otimes \omega_1 + T^1_{12}(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1)\right) \otimes \partial_1 + \left(T^2_{22}\omega_2 \otimes \omega_2 + T^2_{12}(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1)\right) \otimes \partial_2$$

with the components $T^2_{12} = \lambda_1$, $T^1_{12} = \lambda_2$, $T^1_{11} = 2\lambda_1 + \mu$, $T^2_{22} = 2\lambda_2 - \mu$ for some smooth functions λ_1, λ_2 and μ. Here $\{\partial_1, \partial_2\}$ is the dual basis for $\{\omega_1, \omega_2\}$.

Therefore, in order to linearize a 3-web, one should find functions λ_1, λ_2 and μ in such a way that the connection corresponding to $d_T = d_\gamma - T$ is flat.

3. Linearizability conditions for 3-webs

In order to obtain a flat torsion-free connection, components of the affine deformation tensor must satisfy the following *Akivis–Goldberg equations*:

$$\begin{aligned}
&\delta_1(\lambda_1) = \lambda_1(\lambda_1 + \mu),\\
&\delta_2(\lambda_1) = \lambda_1\lambda_2 + \frac{K}{3} + \frac{1}{3}\delta_1(\mu) - \frac{2}{3}\delta_2(\mu),\\
&\delta_1(\lambda_2) = \lambda_1\lambda_2 - \frac{K}{3} + \frac{2}{3}\delta_1(\mu) - \frac{1}{3}\delta_2(\mu),\\
&\delta_2(\lambda_2) = \lambda_2(\lambda_2 - \mu),
\end{aligned} \tag{1}$$

where δ_1 and δ_2 are the covariant derivatives along ∂_1 and ∂_2 with respect to the Chern connection.

We shall look at the above system as a system of partial differential equations with respect to the functions λ_1 and λ_2 provided that μ is given. The compatibility conditions for this system are

$$I_1(\mu) = \delta_{11}(\mu) - 2\delta_{12}(\mu) - \mu\delta_1(\mu) + 2\mu\delta_2(\mu) + \delta_1(K) = 0, \tag{2}$$

$$I_2(\mu) = \delta_{22}(\mu) - 2\delta_{12}(\mu) - 2\mu\delta_1(\mu) + \mu\delta_2(\mu) + \delta_2(K) = 0, \tag{3}$$

where $\delta_{ij} = \frac{1}{2}(\delta_i\delta_j + \delta_j\delta_i)$ are the symmetrized second covariant derivatives.

In order to get a compatible system, the Mayer bracket (see [7]) I_{12} of I_1 and I_2 should be added to (2). In our case

$$\begin{aligned} I_{12} = {} & 24K\mu_{12} + 6(2K_1 - K_2)\mu_1 + 6(2K_2 - K_1)\mu_2 + 24K\mu(\mu_1 - \mu_2) \\ & + 3\mu(K_{11} - K_{12} + K_{22}) - 8K(K_1 + K_2) + 3(K_{112} - K_{122}) - 3K\mu^3, \end{aligned} \tag{4}$$

where μ_i, μ_{ij}, K_i, K_{ij}, K_{ijk} are the symmetrized covariant derivatives of μ and K of the orders 1, 2, etc.

The system $I_1 = 0$, $I_2 = 0$, $I_{12} = 0$ determines a 2-dimensional distribution on the space of 1-jets of μ, and we are looking for 2-dimensional integral manifolds of this distribution. Finally, we get that the system $I_1 = 0$, $I_2 = 0$, $I_{12} = 0$ is compatible if and only if there exists a real and smooth solution of the system of algebraic equations:

$$Q_a = 0, \quad Q_s = 0, \quad Q_{12} = 0, \quad Q_1 = 0, \quad Q_2 = 0, \tag{5}$$

where Q_a, Q_s, Q_{12}, Q_1, Q_2 are the polynomials in μ of degrees 17, 18, 18, 24, and 24, respectively.

We summarize these results in the following theorem:

Theorem 3.1.

- *A planar* 3*-web is linearizable if and only if system* (1) *of differential equations has a smooth solution* $(\lambda_1, \lambda_2, \mu)$.
- *Compatibility conditions for system* (1) *consist of two nonlinear differential equations* (2) *of second order with the unknown function* μ *only.*
- *For the system of differential equations* (2) *to be solvable, one needs to add compatibility condition* (4) *to this system. The compatibility conditions for the resulting system are equivalent to the existence of real and smooth solutions of system* (5) *of algebraic equations.*
- *By substituting a real and smooth root of* (5) *into* (1) *one gets a completely integrable system for functions* (λ_1, λ_2).

4. Differential invariants for linearizability and the Blaschke conjecture

Let $T, S_1, \ldots, S_n$ be polynomials over an algebraically closed field $\mathbb{F}$, $T, S_1, \ldots, S_n \in \mathbb{F}[u]$, and $\operatorname{char}\mathbb{F} = 0$. Denote by $\mathbf{R}(f, g)$ the resultant of polynomials f and g. Then $\mathbf{R}(f, g)$ as a function in g, given f, is homogeneous of degree $\deg f$. Hence $\mathbf{R}(T, x_1S_1 + x_2S_2 + \cdots + x_nS_n)$ is a homogeneous polynomial of degree $\deg T$ in $x_1, \ldots, x_n$.

Let $\mathbf{R}(T, \sum_{i=1}^n x_iS_i) = \sum_\sigma x^\sigma \mathbf{R}_\sigma(T, S_1, \ldots, S_n)$, where σ runs over all multi-indices of the length $\deg T$. We call the coefficients $\mathbf{R}_\sigma(T, S_1, \ldots, S_n)$ *generalized resultants* of the system of polynomials $T, S_1, \ldots, S_n$.

One can prove that *the polynomials* $T, S_1, \ldots, S_n$ *have a common root if and only if all resultants* $R_\sigma(T, S_1, \ldots, S_n)$ *are equal to zero*. Note that the number of resultants $\mathbf{R}_\sigma(T, S_1, \ldots, S_n)$ equals $\binom{n+t-1}{t}$, where $t = \deg T$.

As we indicated earlier, the solvability of the system of differential equations $I_1 = I_2 = I_{12} = 0$ is equivalent to the existence of real roots of the system of algebraic equations (5). This implies that *for a nonparallelizable* 3*-web* W*, the differential invariants* $\mathbf{R}_{i_1i_2i_3i_4}(Q_a, Q_s, Q_{12}, Q_1, Q_2)$ *vanish if and only if algebraic system* (5) *has at least one solution*. Note that all the differential invariants depend on the curvature function K and its covariant derivatives up to order six, but $\mathbf{R}_{i_1 i_2 i_3 i_4}(Q_a, Q_s, Q_{12}, Q_1, Q_2)$ with $i_3 = i_4 = 0$ depend on the curvature function K and its covariant derivatives up to order five. Since for a nonparallelizable 3-web we have $\deg Q_a = 17$, the total number of invariants equals 1040, and among them there are 18 invariants of order five in K. In terms of the web function $f(x, y)$, the corresponding orders are nine and eight.

Note also that the number of invariants is not invariant: it depends on which of the polynomials $Q_a, Q_s, Q_{12}, Q_1, Q_2$ we take as the first one. In our considerations we took the polynomial Q_a of the least degree 17 as the first polynomial. Moreover, the number of invariants can be reduced if we replace Q_a by the linear combination of Q_a and Q_s whose degree is less than 17.

In the book [2] (§17) Blaschke made the following conjecture: the linearizability conditions for a nonparallelizable 3-web are expressed in terms of the web function $f(x, y)$ and its covariant derivatives up to order nine, and the number of differential invariants equals four. As we have seen, Blaschke's estimate of the 'functional codimension' of the orbits of the linearizable 3-webs was correct while the number of algebraic conditions is much greater than four. Moreover, not all linearizability invariants are of order nine: 18 of them are of order eight.

To find out whether algebraic system (5) has real solutions, we consider the greatest common divisor $\mathbf{G} = \mathbf{GCD}[Q_a, Q_s, Q_{12}, Q_1, Q_2]$ of the polynomials $Q_a, Q_s, Q_{12}, Q_1, Q_2$. It is easy to see that *if* $\deg G = 0$, *then there are no common solutions, and the* 3-*web is nonlinearizable*; *if* $\deg G > 1$, *but* G *has no real roots, then the* 3-*web is also nonlinearizable*; *and in the case when* $\deg G = 1$, *or* $\deg G > 1$ *but* G *has a real root, a* 3-*web is linearizable*.

Our main theorem implies that for testing a 3-web W_3 given by a web function $z = f(x, y)$ for linearizability one should compute the curvature K and its covariant derivatives up to order six, find the polynomials $Q_a, Q_s, Q_{12}, Q_1, Q_2$, compute $\mathbf{G} = \mathbf{GCD}[Q_a, Q_s, Q_{12}, Q_1, Q_2]$ and apply the criterion indicated above. We have checked that all the differential invariants vanish for an *arbitrary linear* 3-*web* W_3.

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