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Statistics/Probability Theory

Sums, differences, products, and ratios of hypergeometric beta variables

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Abstract

Recent papers by Professor T. Pham-Gia derived distributions of sums, differences, products and ratios of independent beta random variables. In this Note we extend Professor Pham-Gia's results when X_1 and X_2 are independent random variables distributed according to the confluent and Gauss hypergeometric distributions (which are generalizations of the beta distribution). For each of these distributions, we derive exact expressions for the densities of $S = X_1 + X_2$, $D = X_1 - X_2$, $P = X_1 X_2$, and $R = X_2/X_1$. The expressions turn out to involve the hypergeometric functions of one and two variables. ***To cite this article: S. Nadarajah, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Sommes, différences, produits et rapports de variables hypergéométriques beta. Dans des publications récentes, le Professeur T. Pham-Gia a calculé les distributions de sommes, différences, produits et rapports de variables aléatoires beta indépendantes. Dans cette Note, nous étendons les résultats du Professeur Pham-Gia au cas où X_1 and X_2 sont des variables aléatoires indépendantes obéissant aux distributions hypergéométriques confluente et de Gauss (qui sont des généralisations de la distribution beta). Pour chacune de ces distributions, nous dérivons des expressions excates des densités $S = X_1 + X_2$, $D = X_1 - X_2$, $P = X_1 X_2$ et $R = X_2/X_1$. Ces expressions impliquent des fonctions hypergéométriques de une et deux variables. ***Pour citer cet article : S. Nadarajah, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Pham-Gia pioneered the study of sums, differences, products and ratios of independent beta random variables (see also earlier work by Steece [12], Bhargava and Khatri [2] and Tang and Gupta [13]). Pham-Gia and Turkkan [8,9] and Pham-Gia [7] considered the case when X_i have the standard beta distribution with the probability density function (pdf) given by

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$$f_i(x) = \frac{1}{B(a_i, b_i)} x^{a_i - 1}(1-x)^{b_i - 1} \tag{1}$$

for $0 < x < 1$, $a_i > 0$ and $b_i > 0$, while Pham-Gia and Turkkan [10,11] considered the case of the general beta distribution given by the pdf

$$f_i(x) = \frac{1}{(d_i - c_i)B(a_i, b_i)} \left(\frac{x - c_i}{d_i - c_i}\right)^{a_i - 1} \left(1 - \frac{x - c_i}{d_i - c_i}\right)^{b_i - 1}$$

for $c_i \leqslant x \leqslant d_i$, $a_i > 0$ and $b_i > 0$. See the recent handbook by Gupta and Nadarajah [5] and references therein for the most up-to-date details about the standard beta distribution and its generalizations (see also Johnson et al. [6]).

The aim of this Note is to extend Pham-Gia's work to hypergeometrically distributed random variables. We consider two hypergeometric distributions: the confluent hypergeometric and the Gauss hypergeometric distributions due to Gordy [3] and Armero and Bayarri [1], respectively. The corresponding pdfs are given by

$$f_i(x) = \frac{x^{a_i - 1}(1-x)^{b_i - 1} \exp(-\gamma_i x)}{B(a_i, b_i)\, {}_1F_1(a_i; a_i + b_i; -\gamma_i)} \tag{2}$$

and

$$f_i(x) = \frac{x^{a_i - 1}(1-x)^{b_i - 1}/(1 + z_i x)^{\gamma_i}}{B(a_i, b_i)\, {}_2F_1(\gamma_i, a_i; a_i + b_i; -z_i)}, \tag{3}$$

respectively, for $0 < x < 1$, $a_i > 0$, $b_i > 0$ and $-\infty < \gamma_i < \infty$, where ${}_1F_1$ and ${}_2F_1$ are the confluent and Gauss hypergeometric functions. These hypergeometric distributions have attracted useful applications in many areas including auction theory and traffic modeling, see Armero and Bayarri [1] and Gordy [3]. If $\gamma_i = 0$ then (2) and (3) reduce to the standard beta pdf given by (1). The parameters γ_i are known as the exponential parameters.

We derive explicit expressions for the distributions of $S = X_1 + X_2$, $D = X_1 - X_2$, $P = X_1 X_2$, and $R = X_2/X_1$ when the pdfs f_i are given by (2), (3). These expressions, given in Theorems 1 and 2, involve the special functions ${}_1F_1$ and ${}_2F_1$ as well as the hypergeometric functions Φ_1 and F_1 of two variables. The proofs of the theorems presented in the attached Appendix use properties of these special functions (see, for example, Gradshteyn and Ryzhik [4]).

Theorem 1. *If X_i, $i = 1, 2$, are independent random variables with the pdfs f_i, $i = 1, 2$, given by* (2) *then the pdfs of the sum $S = X_1 + X_2$, the difference $D = X_1 - X_2$, the product $P = X_1 X_2$, and the ratio $R = X_2/X_1$ can be expressed as*

$$f_S(s) = \begin{cases} C s^{a_1 + a_2 - 1}(1-s)^{b_2 - 1} \exp(-\gamma_2 s) \displaystyle\sum_{k=0}^{\infty} \frac{(1 - b_1)_k}{k!} s^k B(a_1 + k, a_2) \\ \quad \times \Phi_1\left(a_1 + k, 1 - b_2, a_1 + a_2 + k; \dfrac{s}{s-1}, (\gamma_2 - \gamma_1)s\right), & \text{if } 0 \leqslant s \leqslant 1, \\ C(s-1)^{a_1 - 1}(2-s)^{b_1 + b_2 - 1} \exp\left\{(1-s)\gamma_1 - \gamma_2\right\} \displaystyle\sum_{k=0}^{\infty} \frac{(1 - a_2)_k}{k!} (2-s)^k \\ \quad \times B(b_1, b_2 + k) \Phi_1\left(b_2 + k, 1 - a_1, b_1 + b_2 + k; \dfrac{s-2}{s-1}, (\gamma_2 - \gamma_1)(2-s)\right), & \text{if } s > 1, \end{cases} \tag{4}$$

$$f_D(d)=\begin{cases} C(-d)^{a_2-1}(1+d)^{a_1+b_2-1}\exp(\gamma_2 d)\displaystyle\sum_{k=0}^{\infty}\frac{(1-b_1)_k}{k!}(1+d)^k B(a_1+k,b_2) \\ \quad\times\Phi_1\left(a_1+k,1-a_2,a_1+b_2+k;\dfrac{1+d}{d},-(\gamma_1+\gamma_2)(1+d)\right), & \text{if } -1\leqslant d\leqslant 0,\\ Cd^{a_1-1}(1-d)^{b_1+a_2-1}\exp(-\gamma_1 d)\displaystyle\sum_{k=0}^{\infty}\frac{(1-b_2)_k}{k!}(1-d)^k B(b_1,a_2+k) \\ \quad\times\Phi_1\left(a_2+k,1-a_1,b_1+a_2+k;\dfrac{d-1}{d},-(\gamma_1+\gamma_2)(1-d)\right), & \text{if } 0\leqslant d\leqslant 1,\end{cases} \tag{5}$$

$$f_P(p)=CB(b_1,b_2)p^{a_1-b_2-1}(1-p)^{b_1+b_2-1}\exp(-\gamma_1 p) \times\sum_{k=0}^{\infty}\frac{(-\gamma_2)^k}{k!}\Phi_1\left(b_2,a_2+b_2-a_1+k,b_1+b_2;\frac{p-1}{p},-\gamma_1(1-p)\right), \tag{6}$$

and

$$f_R(r)=\begin{cases} CB(a_1+a_2,b_1)r^{a_2-1}\Phi_1\big(a_1+a_2,1-b_2,a_1+b_1+a_2;r,-(\gamma_1+\gamma_2 r)\big), & \text{if } r\leqslant 1,\\ CB(a_1+a_2,b_2)r^{-(1+a_1)}\Phi_1\left(a_1+a_2,1-b_1,a_2+b_2+a_1;\dfrac{1}{r},-\left(\dfrac{\gamma_1}{r}+\gamma_2\right)\right), & \text{if } r>1,\end{cases} \tag{7}$$

respectively, for $0\leqslant s\leqslant 2$, $-1\leqslant d\leqslant 1$, $0\leqslant p\leqslant 1$ and $0\leqslant r<\infty$, where the constant C is given by

$$\frac{1}{C}=B(a_1,b_1)\,{}_1F_1(a_1;a_1+b_1;-\gamma_1)B(a_2,b_2)\,{}_1F_1(a_2;a_2+b_2;-\gamma_2).$$

Theorem 2. *If X_i, $i=1,2$, are independent random variables with the pdfs f_i, $i=1,2$, given by* (3) *then the pdfs of the sum $S=X_1+X_2$, the difference $D=X_1-X_2$, the product $P=X_1X_2$, and the ratio $R=X_2/X_1$ can be expressed as*

$$f_X(x)=C\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{k+l}(\gamma_1)_k(\gamma_2)_l z_1^k z_2^l}{k!l!}I_{k,l}(x) \tag{8}$$

for $(x,X)=(s,S)$, (d,D), (p,P), (r,R), $0\leqslant s\leqslant 2$, $-1\leqslant d\leqslant 1$, $0\leqslant p\leqslant 1$ and $0\leqslant r<\infty$, where

$$I_{k,l}(s)=\begin{cases} B(a_1+k,a_2+l)s^{a_1+a_2+k+l-1}(1-s)^{b_2-1} \\ \quad\times F_1\left(a_1+k,1-b_1,1-b_2,a_1+a_2+k+l;s,\dfrac{s}{s-1}\right), & \text{if } 0\leqslant s\leqslant 1,\\ B(b_1,b_2)(s-1)^{a_1+k-1}(2-s)^{b_1+b_2-1} \\ \quad\times F_1\left(b_2,1-a_2-l,1-a_1-k,b_1+b_2;2-s,\dfrac{2-s}{1-s}\right), & \text{if } s>1,\end{cases} \tag{9}$$

$$I_{k,l}(d)=\begin{cases} B(a_1+k,b_2)(-d)^{a_2+l-1}(1+d)^{a_1+b_2+k-1} \\ \quad\times F_1\left(a_1+k,1-b_1,1-a_2-l,a_1+b_2+k;1+d,\dfrac{1+d}{d}\right), & \text{if } -1\leqslant d\leqslant 0,\\ B(b_1,a_2+l)d^{a_1+k-1}(1-d)^{a_2+b_1+l-1} \\ \quad\times F_1\left(a_2+l,1-b_2,1-a_1-k,a_2+b_1+l;1-d,\dfrac{d-1}{d}\right), & \text{if } 0\leqslant d\leqslant 1,\end{cases} \tag{10}$$

$$I_{k,l}(p)=B(b_1,b_2)p^{a_1-b_2+k-1}(1-p)^{b_1+b_2-1}\,{}_2F_1\left(a_2+b_2-a_1+l-k,b_2;b_1+b_2;\frac{p-1}{p}\right), \tag{11}$$

$$I_{k,l}(r) = \begin{cases} B(a_1 + a_2 + k + l, b_1) r^{a_2 + l - 1} \\ \quad \times {}_2F_1(1 - b_2, a_1 + a_2 + k + l; a_1 + b_1 + a_2 + k + l; r), & \text{if } r \leqslant 1, \\ B(a_1 + a_2 + k + l, b_1) r^{-(a_1 + k + b_1)} \\ \quad \times {}_2F_1(1 - b_2, a_1 + a_2 + k + l; a_2 + b_1 + a_1 + k + l; 1), & \text{if } r > 1, \end{cases} \tag{12}$$

and the constant C is given by

$$\frac{1}{C} = B(a_1, b_1)\, {}_2F_1(\gamma_1, a_1; a_1 + b_1; -z_1) B(a_2, b_2)\, {}_2F_1(\gamma_2, a_2; a_2 + b_2; -z_2).$$

The formulas in Theorems 1 and 2 have been checked for accuracy by computing them (numerically) using MAPLE and comparing with the values of the densities obtained directly by numerical integration.

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