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Number Theory

Fractional parts of powers and Sturmian words ☆

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Abstract

Let $b \geqslant 2$ be an integer. In terms of combinatorics on words we describe all irrational numbers $\xi > 0$ with the property that the fractional parts $\{\xi b^n\}$, $n \geqslant 0$, all belong to a semi-open or an open interval of length $1/b$. The length of such an interval cannot be smaller, that is, for irrational ξ, the fractional parts $\{\xi b^n\}$, $n \geqslant 0$, cannot all belong to an interval of length smaller than $1/b$. ***To cite this article: Y. Bugeaud, A. Dubickas, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Parties fractionnaires de puissances et mots sturmiens. Soit $b \geqslant 2$ un entier. Au moyen de résultats de la combinatoire des mots, nous caractérisons l'ensemble des nombres réels $\xi > 0$ tels que les parties fractionnaires $\{\xi b^n\}$, $n \geqslant 0$, appartiennent toutes à un intervalle semi-ouvert ou ouvert de longueur $1/b$. La longueur d'un tel intervalle ne peut pas être plus petite, c'est-à-dire, quel que soit le nombre irrationnel ξ, aucun intervalle de longueur strictement inférieure à $1/b$ ne contient toutes les parties fractionnaires $\{\xi b^n\}$, $n \geqslant 0$. ***Pour citer cet article : Y. Bugeaud, A. Dubickas, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Version française abrégée

Dans tout ce qui suit, $\{\cdot\}$ désigne la fonction partie fractionnaire. Suivant la définition énoncée en 1968 par Mahler [11], un Z-nombre est un nombre réel positif ξ vérifiant $0 \leqslant \{\xi(3/2)^n\} < 1/2$ pour tout entier $n \geqslant 0$. L'ensemble des Z-nombres est au plus dénombrable [11] et même vraisemblablement vide, mais ce problème difficile n'est à ce jour pas résolu. Plus généralement, étant donnés un nombre réel $\alpha > 1$ et un sous-intervalle $[s, t[$

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de $[0,1[$, on souhaiterait savoir s'il existe un nombre réel $\xi > 0$ vérifiant $s \leqslant \{\xi\alpha^n\} < t$ pour tout entier $n \geqslant 0$, ou bien, plus modestement, on aimerait déterminer la plus petite longueur $t - s$ pour laquelle un tel ξ existe.

Cette Note répond à deux objectifs : nous annonçons des résultats nouveaux obtenus pour α algébrique par le second auteur et nous apportons une réponse complète aux questions *supra* lorsque $\alpha \geqslant 2$ est un entier.

Soient p et q des entiers vérifiant $p > q \geqslant 2$. Flatto, Lagarias et Pollington [9] établirent que, pour tout intervalle I de longueur strictement inférieure à $1/p$, il n'existe aucun nombre réel $\xi > 0$ vérifiant $\{\xi(p/q)^n\} \in I$ pour tout entier $n \geqslant 0$ (cf. également [3]). Une nouvelle démonstration, plus simple, de ce résultat, ainsi que sa généralisation aux nombres algébriques réels > 1 qui ne sont ni de Pisot, ni de Salem, se trouvent dans deux travaux récents [4,6] du second auteur.

Théorème 0.1 [4,6]. *Soit $\alpha > 1$ un nombre réel algébrique, et soit $P(X) \in \mathbf{Z}[X]$ son polynôme minimal. Soit $F(X)$ un polynôme à coefficients réels, de degré $r \geqslant 0$, et dont le coefficient dominant est positif. Supposons en outre que $F(X) \notin \mathbf{Q}(\alpha)[X]$ si ou bien α est un nombre de Pisot, ou bien $r = 0$ et α est un nombre de Salem. Alors, les parties fractionnaires $\{F(n)\alpha^n\}$, $n \geqslant 0$, ne peuvent pas toutes se trouver dans un intervalle de longueur strictement inférieure à $1/\ell(P^{r+1})$.*

Ici, $\ell(P^{r+1})$ désigne la *longueur réduite* du polynôme $P(X)^{r+1}$, définie *infra*. Les hypothèses sur le polynôme $F(X)$ sont nécessaires [14].

Nous supposons désormais que α est un entier > 1, et choisissons de le noter b. Il découle du Théorème 0.1 que, pour tout *irrationnel* ξ, la longueur de tout intervalle I contenant toutes les parties fractionnaires $\{\xi b^n\}$, $n \geqslant 0$, est au moins égale à $1/b$. Dans cette note, nous caractérisons complètement les paires (ξ, I), formées d'un nombre réel irrationnel $\xi > 0$ et d'un intervalle I, pour lesquelles $\{\xi b^n\}$ appartient à I pour tout $n \geqslant 0$. Nous employons la terminologie de la combinatoire des mots [2,10], et notamment la notion de suite sturmienne.

Théorème 0.2. *Soient $b \geqslant 2$ un entier et ξ un nombre réel irrationnel. Les parties fractionnaires $\{\xi b^n\}$, $n \geqslant 0$, ne peuvent pas toutes se trouver dans un intervalle de longueur strictement inférieure à $1/b$. En outre, les nombres $\{\xi b^n\}$, $n \geqslant 0$, sont tous dans un intervalle fermé I de longueur $1/b$ si, et seulement si, $\xi = g + k/(b-1) + t_b(\mathbf{w})$, où g est un entier quelconque, k appartient à $\{0, 1, \ldots, b-2\}$ et $\mathbf{w}$ est un mot sturmien sur $\{0, 1\}$. Si tel est le cas, alors ξ est transcendant et l'intervalle I est semi-ouvert. De plus, I est ouvert sauf s'il existe un entier $j \geqslant 0$ et un mot sturmien caractéristique $\mathbf{u}$ tels que $T^j(\mathbf{w}) = \mathbf{u}$.*

En particulier, puisqu'il existe une infinité non dénombrable de suites sturmiennes sur $\{0, 1\}$, le Théorème 0.2 montre qu'il existe une infinité non dénombrable de paires (ξ, s), où ξ est irrationnel et $s \in]0, 1 - 1/b[$, telles que $s < \{\xi b^n\} < s + 1/b$ pour tout $n \geqslant 0$.

1. Introduction

In 1968, Mahler [11] introduced the notion of Z-numbers. These are precisely the positive real numbers ξ such that $0 \leqslant \{\xi(3/2)^n\} < 1/2$ for all integers $n \geqslant 0$. Here and below, $\{\cdot\}$ denotes the fractional part. The set of Z-numbers is at most countable [11], and it is widely believed it is even empty. This raises the following more general questions. Given a real number $\alpha > 1$ and an interval $[s, t)$ included in $[0, 1)$, are there any positive numbers ξ such that $s \leqslant \{\xi\alpha^n\} < t$ for all integers $n \geqslant 0$? What is the smallest possible difference $t - s$ for which such positive numbers ξ do exist?

The purpose of this note is twofold. Firstly, we announce several new results obtained by the second named author for algebraic numbers α. Secondly, we give a complete answer to the above questions for rational integers $\alpha = b \geqslant 2$.

Let p and q be coprime integers with $p > q \geqslant 2$. Flatto, Lagarias and Pollington [9] showed that, for any interval I of length strictly smaller than $1/p$, there are no $\xi > 0$ such that $\{\xi(p/q)^n\} \in I$ for all integers $n \geqslant 0$. Presumably, this also holds for any interval $I = [s, s + 1/p)$, with $s \in [0, 1 - 1/p]$. Actually, it was proved in [9] that this is the case for any s lying in a dense subset of $[0, 1 - 1/p]$, and, later, the first named author established [3] that this is also the case for any s lying in a subset of full Lebesgue measure of $[0, 1 - 1/p]$.

A new, simpler proof of the result of Flatto, Lagarias and Pollington and its generalization from rational non-integer numbers $\alpha = p/q$ to arbitrary real algebraic numbers α which are neither PV-numbers nor Salem numbers has been recently given by the second named author [4,6]. Recall that an algebraic integer $\alpha > 1$ is called a *PV-number* (resp. *Salem number*) if its remaining conjugates (if any) are all inside the unit disc $|z| < 1$ (resp. in $|z| \leqslant 1$ with at least one conjugate lying on $|z| = 1$). To state these results, we define the *reduced length* of a polynomial $P(X) \in \mathbf{R}[X]$, denoted by $\ell(P)$, to be the infimum of the lengths (that is, the sums of the absolute values of the coefficients) of the polynomials $P(X) \cdot G(X)$, taken over every polynomial $G(X) \in \mathbf{R}[X]$ with either leading coefficient or constant coefficient equal to 1. It is easy to prove that $\ell(qX - p) = p$ for all integers $p > q \geqslant 1$ (see [4] or [13], where the reduced length of a polynomial was studied in detail).

Theorem 1.1 [4,6]. *Let $\alpha > 1$ be a real algebraic number with minimal defining polynomial $P(X) \in \mathbf{Z}[X]$, and let $F(X)$ be a degree $r \geqslant 0$ real polynomial with positive leading coefficient. Suppose, in addition, that $F(X) \notin \mathbf{Q}(\alpha)[X]$ if either α is a PV-number or $r = 0$ and α is a Salem number. Then the fractional parts $\{F(n)\alpha^n\}$, $n \geqslant 0$, cannot all lie in an interval of length smaller than $1/\ell(P^{r+1})$.*

The extra conditions on $F(X)$ in Theorem 1.1 that concern PV and Salem numbers α are necessary. This is clear for PV-numbers α, whereas for Salem numbers α the necessity of the condition $\xi = F(X) \notin \mathbf{Q}(\alpha)$, where $r = \deg F = 0$, is shown in [14]. Other results related to Theorem 1.1 have been obtained in [1] and [5].

From now on, suppose that $\alpha > 1$ is an integer, say $\alpha = b \geqslant 2$. It is a PV-number, so Theorem 1.1 implies that, for any interval I of length strictly smaller than $1/b$, there are no *irrational* numbers ξ for which $\{\xi b^n\} \in I$ for every integer $n \geqslant 0$. In particular, it follows from this that $1/b$ is the smallest possible length of an interval to which all the fractional parts $\{\xi b^n\}$, $n \geqslant 0$, with fixed irrational ξ, can belong. This also raises the question whether, for an interval I of length $1/b$, there exists an irrational number ξ such that $\{\xi b^n\} \in I$ for all integers $n \geqslant 0$. In the present note, we show that there are uncountably many pairs (ξ, I) with this property and describe all of them.

Note that, writing the b-adic expansion of $\{\xi\}$, namely, $\xi = g + x_1 b^{-1} + x_2 b^{-2} + \cdots$, where $g = [\xi]$ and $x_1, x_2, \ldots \in \{0, 1, \ldots, b - 1\}$, we have

$$\{\xi b^n\} = x_{n+1} b^{-1} + x_{n+2} b^{-2} + x_{n+2} b^{-3} + \cdots := 0.x_{n+1} x_{n+2} x_{n+3} \cdots$$

for any $n \geqslant 0$. So, in other words, we are interested in the following question: determine the smallest possible interval I to which belong all the *tails* of an irrational number $\xi = g + 0.x_1 x_2 x_3 \cdots$ (in its b-adic expansion), namely, the numbers $0.x_{n+1} x_{n+2} x_{n+2} \cdots$, where $n \geqslant 0$.

2. Main result

We will use the terminology from combinatorics on words (see, for instance, [2] or [10]). For an infinite word $\mathbf{w}$, let us denote by $p(\mathbf{w}, m)$ the number of distinct blocks of length m occurring in $\mathbf{w}$. Morse and Hedlund [12] proved that the function $m \mapsto p(\mathbf{w}, m)$ is either bounded, or strictly increasing. Consequently, $\mathbf{w}$ is not ultimately periodic (in this context usually called *aperiodic*) if, and only if, $p(\mathbf{w}, m) \geqslant m + 1$ holds for every positive integer m. By definition, an infinite word $\mathbf{w}$ is called *Sturmian* if we have $p(\mathbf{w}, m) = m + 1$ for any positive integer m. (In particular, since then $p(\mathbf{w}, 1) = 2$, this implies that $\mathbf{w}$ is a word on an alphabet of two letters.) There are many equivalent definitions for Sturmian words, and we refer the reader to Chapter 2 from [10] or to Chapter 6 from [2]. We just recall that $\mathbf{w} := w_1 w_2 \cdots$ is a *characteristic Sturmian word* if, and only if, there exists an irrational number β (the *slope*) in $(0, 1)$ such that $w_n = [\beta(n+1)] - [\beta n]$ for every positive integer n.

Suppose that T^j maps the word $\mathbf{w} = w_1 \cdots w_j w_{j+1} \cdots$ to the word $w_{j+1} w_{j+2} \cdots$ and set $t_b(\mathbf{w}) := 0.w_1 w_2 \cdots = \sum_{j=1}^{\infty} w_j b^{-j}$. With this notation, we can state our main result.

Theorem 2.1. *Let $b \geqslant 2$ be an integer and ξ be an irrational real number. Then the numbers $\{\xi b^n\}$, $n \geqslant 0$, cannot all lie in an interval of length strictly smaller than $1/b$. On the other hand, the real numbers $\{\xi b^n\}$, $n \geqslant 0$, are all lying in a closed interval I of length $1/b$ if, and only if, $\xi = g + k/(b-1) + t_b(\mathbf{w})$, where g is an arbitrary integer, k is in $\{0, 1, \ldots, b-2\}$, and $\mathbf{w}$ is a Sturmian word on $\{0, 1\}$. If this is the case, then ξ is transcendental and the interval I is semi-open. Moreover, it is open, unless there exists an integer $j \geqslant 1$ such that $T^j(\mathbf{w}) = \mathbf{u}$ is a characteristic Sturmian word.*

In particular, since there are uncountably many Sturmian sequences on $\{0, 1\}$, Theorem 2.1 shows that there are uncountably many pairs (ξ, s), where ξ is irrational and $s \in (0, 1 - 1/b)$, such that $s < \{\xi b^n\} < s + 1/b$ for every $n \geqslant 0$.

At the end of the paper [7] the following problem is posed: prove that, for any real numbers ξ and ν with $\xi > 0$, the numbers $[\xi 2^n + \nu]$ are composite for infinitely many $n \in \mathbf{N}$. Observe that if we have $0 \leqslant \{\xi 2^{n-1} + (\nu - 1)/2\} < 1/2$, then the number $[\xi 2^n + \nu - 1]$ is even and so $[\xi 2^n + \nu]$ is odd. Thus, since there are uncountably many Sturmian words on the alphabet $\{0, 1\}$, it follows from Theorem 2.1 that there do exist uncountably many pairs (ξ, ν) for which $[\xi 2^n + \nu]$ is odd for every positive integer n.

3. Proof of Theorem 2.1

Before giving the proof of Theorem 2.1, we gather in an auxiliary lemma results from Proposition 2.1.3, Theorem 2.1.5 and Proposition 2.1.22 of Chapter 2 of [10].

Lemma 3.1. *Let $\mathbf{w}$ be an infinite aperiodic word on $\{0, 1\}$. Then, $\mathbf{w}$ is Sturmian if, and only if, for any finite word $\mathbf{v}$, at least one of the words $0\mathbf{v}0$ and $1\mathbf{v}1$ is not a factor of $\mathbf{w}$. Moreover, $\mathbf{w}$ is Sturmian characteristic if, and only if, both $0\mathbf{w}$ and $1\mathbf{w}$ are Sturmian.*

Let us write ξ in the form $g + t_b(\mathbf{x})$, where $g = [\xi]$ is an integer and $t_b(\mathbf{x}) = x_1 b^{-1} + x_2 b^{-2} + x_3 b^{-3} + \cdots = 0.x_1 x_2 x_3 \cdots$ is the b-adic expansion of $\{\xi\} = \xi - g$. As above, $\{\xi b^n\} = 0.x_{n+1} x_{n+2} \cdots$. In particular, since ξ is irrational, this implies that $x_{n+1} b^{-1} < \{\xi b^n\} < x_{n+1} b^{-1} + b^{-1}$. Thus, if there exist x_{i+1} and x_{j+1}, with $i, j \geqslant 0$, satisfying $x_{j+1} - x_{i+1} \geqslant 2$, then we get

$$\{\xi b^j\} - \{\xi b^i\} > x_{j+1} b^{-1} - x_{i+1} b^{-1} - b^{-1} \geqslant 2/b - 1/b = 1/b.$$

Consequently, we can assume without loss of generality that $x_1, x_2, \ldots \in \{k, k+1\}$, where $k = 0, 1, \ldots, b-2$. Thus, we can write ξ in the form $g + k/(b-1) + t_b(\mathbf{w})$, where $\mathbf{w} = w_1 w_2 \cdots$ is a word on the alphabet $\{0, 1\}$ and $t_b(\mathbf{w}) = w_1 b^{-1} + w_2 b^{-2} + w_3 b^{-3} + \cdots = 0.w_1 w_2 w_3 \cdots$. Now, we have

$$\{\xi b^n\} - k/(b-1) = 0.w_{n+1} w_{n+2} \cdots = w_{n+1} b^{-1} + w_{n+2} b^{-2} + \cdots.$$

Since ξ is irrational, the complexity function of the infinite word $\mathbf{w} := w_1 w_2 \cdots$ is strictly increasing. This implies that, for any $m \geqslant 1$, there exists (at least) one block $\mathbf{w}_m$ of m letters such that both $0\mathbf{w}_m$ and $1\mathbf{w}_m$ are subblocks of $\mathbf{w}$. In other words, there exist integers $u = u(m)$ and $v = v(m)$ such that $\{\xi b^u\} - k/(b-1) = 0.0\mathbf{w}_m\mathbf{w}'$ and $\{\xi b^v\} - k/(b-1) = 0.1\mathbf{w}_m\mathbf{w}''$. Hence $\{\xi b^v\} - \{\xi b^u\} > b^{-1} - b^{-m}$. By taking m sufficiently large, we conclude that no interval of length strictly smaller than $1/b$ can contain all the $\{\xi b^n\}$ with $n \geqslant 0$. (Taking $\xi = 0.101010\ldots$ or simply $\xi = 1$ shows that the assumption 'ξ is irrational' is necessary.)

Let us now prove the second statement. Assume that **w** is Sturmian. By Lemma 3.1, for any finite word **v**, the words 0**v**0 and 1**v**1 cannot be both factors of **w**. Consequently, the difference between any two numbers $\{\xi b^j\}$ and $\{\xi b^i\}$ is bounded above in absolute value by $1/b$. The inequality is strict, since **w** is aperiodic. Thus, we have shown that, for **w** Sturmian, there exists a semi-open interval of length $1/b$ that contains all the $\{\xi b^n\}$, where $n \geqslant 0$. Furthermore, it follows from [8] that ξ is transcendental.

Assume now that **w** is neither Sturmian, nor ultimately periodic. Then, by the lemma, there exists a finite world **u** such that both 0**u**0 and 1**u**1 are factors of **w**. Arguing as above, we see that the difference between corresponding fractional parts is greater than $1/b$. This shows that, for such w, there does not exist a closed interval of length $1/b$ containing all fractional parts $\{\xi b^n\}$, $n \geqslant 0$, and proves the second part.

Finally, let **w** be an infinite Sturmian word. The fact that the numbers $0.w_{n+1}w_{n+2}\cdots$ all belong to a closed interval of length $1/b$ can be expressed in the form

$$t_b(0\mathbf{u}) \leqslant t_b(T^n\mathbf{w}) \leqslant t_b(1\mathbf{u}) = t_b(0\mathbf{u}) + b^{-1},$$

where **u** is a word on $\{0, 1\}$ and where n runs through every non-negative integer. For simplicity (and according to the lexicographical order of words), we can write this inequality in the form

$$0\mathbf{u} \leqslant T^n\mathbf{w} \leqslant 1\mathbf{u}, \quad \text{for any } n \geqslant 0.$$

Evidently, all $t_b(T^n\mathbf{w})$ belong to an open interval of length $1/b$, unless there is a $h \geqslant 0$ such that $T^h\mathbf{w} = 0\mathbf{u}$ or $1\mathbf{u}$. Assume that $T^h\mathbf{w} = 0\mathbf{u}$. Then we have

$$0\mathbf{u} < T^n\mathbf{w} < 1\mathbf{u}, \quad \text{for any } n \geqslant h+1,$$

that is,

$$0\mathbf{u} < T^n\mathbf{u} < 1\mathbf{u}, \quad \text{for any } n \geqslant 0.$$

The case $T^h\mathbf{w} = 1\mathbf{u}$ leads to the same inequalities. These are strict, since **w** is aperiodic.

Let us prove now that **u** is Sturmian characteristic. In view of Lemma 3.1, it is sufficient to show that both 0**u** and 1**u** are Sturmian. Observe that **u** is aperiodic, since $\mathbf{u} = T^{h+1}(\mathbf{w})$. Assume that $p(0\mathbf{u}, m) \geqslant m+2$ for some m. The first part of Theorem 2.1 implies that 0**u** is a limit point of the sequence $\mathbf{u}, T^1\mathbf{u}, T^2\mathbf{u}, \ldots$, hence, we get that $p(T^n\mathbf{u}, m) \geqslant m+2$ for some n. This yields

$$m+1 = p(\mathbf{w}, m) \geqslant p(T^n\mathbf{u}, m) \geqslant m+2,$$

a contradiction. Consequently, 0**u** is Sturmian, and so is 1**u**, by a similar argument.

We thus conclude that the numbers $t_b(T^n\mathbf{w})$, $n \geqslant 0$, all belong to an open interval of length $1/b$, unless there are an integer $h \geqslant 0$ and a characteristic Sturmian word **u** such that $T^h\mathbf{w} = 0\mathbf{u}$ or $1\mathbf{u}$. So $\mathbf{u} = T^j(\mathbf{w})$ with $j = h+1 \geqslant 1$. The proof of Theorem 2.1 is completed.

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