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Partial Differential Equations

On a Liouville-type comparison principle for solutions of semilinear elliptic partial differential inequalities

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Abstract

This Note is devoted to the study of a Liouville-type comparison principle for entire weak solutions of semilinear elliptic partial differential inequalities of the form $Lu + |u|^{q-1}u \leqslant Lv + |v|^{q-1}v$, where $q > 0$ is a given number and L is a linear (possibly non-uniformly) elliptic partial differential operator of second order in divergent form given formally by the relation

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ij}(x)\frac{\partial}{\partial x_j}\right].$$

We assume that $n \geqslant 2$, that the coefficients $a_{ij}(x)$, $i, j = 1, \dots, n$, are measurable bounded functions on $\mathbb{R}^n$ such that $a_{ij}(x) = a_{ji}(x)$, and that the corresponding quadratic form is non-negative. The results obtained in this work complete similar results on solutions of quasilinear elliptic partial differential inequalities announced in Kurta [C. R. Acad. Sci. Paris, Ser. I 336 (11) (2003) 897–900]. ***To cite this article: V.V. Kurta, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Sur un principe de comparaison de type Liouville pour des solutions d'inégalités aux dérivées partielles elliptiques semi-linéaires. Cette Note est consacré à l'étude d'un principe de comparaison de type Liouville pour des solutions entières faibles d'inégalités aux derivées partielles elliptiques semi-linéaires de la forme $Lu + |u|^{q-1}u \leqslant Lv + |v|^{q-1}v$, où $q > 0$ est un nombre donné et L un opérateur aux dérivées partielles (possiblement non-uniformément) elliptique linéaire de deuxième ordre en forme divergente donné formellement par la relation

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ij}(x)\frac{\partial}{\partial x_j}\right].$$

Nous supposons que $n \geqslant 2$, que les coefficients $A_{ij}(x)$, $i, j = 1, \dots, n$, sont des fonctions bornées mesurables en $\mathbb{R}^n$ telles que $a_{ij}(x) = a_{ij}(x)$, et que la forme quadratique correspondante est non-négative. Les résultats obtenus dans ce travail complètent

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d'autres résultats similaires pour des solutions d'inégalités aux dérivées partielles elliptiques announcés dans Kurta [C. R. Acad. Sci. Paris, Ser. I 336 (11) (2003) 897–900]. ***Pour citer cet article : V.V. Kurta, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

1. Introduction and preliminaries

This Note is devoted to the study of a Liouville-type comparison principle for entire weak solutions of semilinear elliptic partial differential inequalities of the form

$$Lu + |u|^{q-1}u \leqslant Lv + |v|^{q-1}v, \tag{1}$$

where $q > 0$ is a given number and L is a linear (possibly non-uniformly) elliptic partial differential operator of second order in divergent form given formally by

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ij}(x)\frac{\partial}{\partial x_j}\right]. \tag{2}$$

Here and in what follows, we assume that $n \geqslant 2$, that the coefficients $a_{ij}(x)$, $i, j = 1, \ldots, n$, are measurable bounded functions on $\mathbb{R}^n$ such that $a_{ij}(x) = a_{ji}(x)$, and that the corresponding quadratic form is non-negative, i.e.,

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant 0 \tag{3}$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ at almost all $x \in \mathbb{R}^n$.

Remark 1. It is important to note that if u and v satisfy, respectively, the inequalities

$$-Lu \geqslant |u|^{q-1}u \tag{4}$$

and

$$-Lv \leqslant |v|^{q-1}v, \tag{5}$$

then the pair (u, v) satisfies inequality (1). Thus, all the results obtained in this work for solutions of (1) are valid for the corresponding solutions of the system (4), (5).

Remark 2. The results obtained in this work complete similar results on solutions of quasilinear elliptic partial differential inequalities announced in [1]. To prove these results we further develop the approach that was proposed for solving similar problems in wide classes of partial differential equations and inequalities in [2].

Definition 1.1. Let $q > 0$. By an entire weak solution of inequality (1) we understand a pair (u, v) of functions $u(x)$ and $v(x)$ measurable on $\mathbb{R}^n$ which belong to the space $W^1_{2,\mathrm{loc}}(\mathbb{R}^n) \cap L_{q,\mathrm{loc}}(\mathbb{R}^n)$ and satisfy the integral inequality

$$\int_{\mathbb{R}^n}\left[\sum_{i,j=1}^{n} a_{ij}u_{x_i}\varphi_{x_j} - |u|^{q-1}u\varphi\right]\mathrm{d}x \geqslant \int_{\mathbb{R}^n}\left[\sum_{i,j=1}^{n} a_{ij}v_{x_i}\varphi_{x_j} - |v|^{q-1}v\varphi\right]\mathrm{d}x \tag{6}$$

for every non-negative function $\varphi \in C^\infty(\mathbb{R}^n)$ with compact support.

Analogous definitions for solutions of inequalities (4) and (5), which are special cases of inequality (1) for $v = 0$ and $u = 0$, respectively, can be immediately obtained from Definition 1.1.

Definition 1.2. Let $q > 0$. By an entire weak solution of inequality (4) (resp., (5)) we understand a function $w(x)$ measurable on $\mathbb{R}^n$ which belongs to the space $W^1_{2,\mathrm{loc}}(\mathbb{R}^n) \cap L_{q,\mathrm{loc}}(\mathbb{R}^n)$ and satisfies the integral inequality

$$\int\limits_{\mathbb{R}^n} \sum_{i,j=1}^{n} a_{ij} w_{x_i} \varphi_{x_j}\,\mathrm{d}x - \int\limits_{\mathbb{R}^n} |w|^{q-1} w\varphi\,\mathrm{d}x \geqslant 0 \quad (\text{resp., } \leqslant 0) \tag{7}$$

for every non-negative function $\varphi \in C^\infty(\mathbb{R}^n)$ with compact support.

2. Main results

Theorem 2.1. *Let $n = 2$, $q > 0$, and (u, v) be an entire weak solution of inequality* (1) *on $\mathbb{R}^n$ such that $u(x) \geqslant v(x)$. Then $u(x) = v(x)$ on $\mathbb{R}^n$.*

Theorem 2.2. *Let $n > 2$, $1 < q \leqslant \frac{n}{n-2}$, and (u, v) be an entire weak solution of inequality* (1) *on $\mathbb{R}^n$ such that $u(x) \geqslant v(x)$. Then $u(x) = v(x)$ on $\mathbb{R}^n$.*

Theorem 2.3. *Let $n > 2$ and $q > \frac{n}{n-2}$. Then there exists no entire weak solution (u, v) of inequality* (1) *on $\mathbb{R}^n$ such that $u(x) \geqslant v(x)$ and the relation*

$$\limsup_{R\to+\infty} R^{-n+2(q-\nu)/(q-1)} \int\limits_{|x|<R} \big(u(x) - v(x)\big)^{q-\nu}\,\mathrm{d}x = +\infty \tag{8}$$

holds with any given $\nu \in (0, 1)$.

Example 1. To illustrate the sharpness of Theorem 2.3 we note that for $n > 2$, $q > \frac{n}{n-2}$, and a suitable constant $c > 0$, the pair (u, v) of functions

$$u(x) = c\big(1 + |x|^2\big)^{-1/(q-1)} \quad \text{and} \quad v(x) = 0 \tag{9}$$

is an entire weak solution of inequality (1) on $\mathbb{R}^n$ for $L = \Delta$ such that for any given $\nu \in (0, 1)$ the relation

$$\limsup_{R\to+\infty} R^{-n+2(q-\nu)/(q-1)} \int\limits_{|x|<R} \big(u(x) - v(x)\big)^{q-\nu}\,\mathrm{d}x = C_1, \tag{10}$$

with C_1 a certain positive constant, holds.

The two following statements are simple special cases of Theorem 2.3:

Theorem 2.4. *Let $n > 2$ and $q > \frac{n}{n-2}$. Then for any given constants $c > 0$ and $\nu > 0$ there exists no entire weak solution (u, v) of inequality* (1) *on $\mathbb{R}^n$ such that $u(x) \geqslant v(x) + c(1 + |x|^2)^{-1/(q-1)+\nu}$.*

Theorem 2.5. *Let $n > 2$ and $q > \frac{n}{n-2}$. Then for any given constant $c > 0$ there exists no entire weak solution (u, v) of inequality* (1) *on $\mathbb{R}^n$ such that $u(x) \geqslant v(x) + c$.*

Theorem 2.6. *Let $n > 2$ and $0 < q < 1$. Then there exists no entire weak solution (u, v) of inequality* (1) *on $\mathbb{R}^n$ such that $u(x) > v(x)$ and the relation*

$$\limsup_{R\to+\infty} R^{2-n} \int\limits_{|x|<R} \big(|u|^{q-1}u - |v|^{q-1}v\big)(u - v)^{-1}\,\mathrm{d}x = +\infty \tag{11}$$

holds.

Example 2. To illustrate the sharpness of Theorem 2.6 we note that for $n > 2$, $0 < q < 1$, $0 < \mu < \frac{n}{n-2}$, and a suitable constant $c > 0$, the pair (u, v) of functions

$$u(x) = c\big(1 + |x|^2\big)^{1/(1-q)} + \big(1 + |x|^2\big)^{-\mu} \tag{12}$$

and

$$v(x) = c\big(1 + |x|^2\big)^{1/(1-q)} \tag{13}$$

is an entire weak solution of inequality (1) on $\mathbb{R}^n$ for $L = \Delta$ such that $u(x) > v(x)$ and the relation

$$\limsup_{R\to+\infty} R^{2-n} \int\limits_{|x|<R} \big(|u|^{q-1}u - |v|^{q-1}v\big)(u - v)^{-1}\,\mathrm{d}x = C_2, \tag{14}$$

with C_2 a certain positive constant, holds.

Example 3. We also note that for $n > 2$, $0 < q < 1$, $0 < \mu < \frac{n}{n-2}$, $\lambda > \frac{1}{1-q}$, and a suitable constant $c > 0$, the pair (u, v) of functions

$$u(x) = c\big(1 + |x|^2\big)^{\lambda} + \big(1 + |x|^2\big)^{-\mu} \tag{15}$$

and

$$v(x) = c\big(1 + |x|^2\big)^{\lambda} \tag{16}$$

is an entire weak solution of inequality (1) on $\mathbb{R}^n$ for $L = \Delta$ such that $u(x) > v(x)$ and the relation

$$\limsup_{R\to+\infty} R^{2-n} \int\limits_{|x|<R} \big(|u|^{q-1}u - |v|^{q-1}v\big)(u - v)^{-1}\,\mathrm{d}x = 0 \tag{17}$$

holds.

Example 4. To illustrate the importance of condition (11) in Theorem 2.6 we note that for $n > 2$, $0 < q < 1$, $0 < \mu < \frac{n}{n-2}$, $\lambda \geqslant \frac{1+\mu}{1-q}$, a suitable constant $c > 0$, and any given constant $\kappa > 0$, the pair (u, v) of functions

$$u(x) = c\big(1 + |x|^2\big)^{\lambda} + \kappa + \big(1 + |x|^2\big)^{-\mu} \tag{18}$$

and

$$v(x) = c\big(1 + |x|^2\big)^{\lambda} \tag{19}$$

is an entire weak solution of inequality (1) on $\mathbb{R}^n$ for $L = \Delta$ such that $u(x) > v(x) + \kappa$ and the relation

$$\limsup_{R\to+\infty} R^{2-n} \int\limits_{|x|<R} \big(|u|^{q-1}u - |v|^{q-1}v\big)(u - v)^{-1}\,\mathrm{d}x = 0 \tag{20}$$

holds.

Theorem 2.7. *Let $n > 2$ and $q = 1$. Then there exists no entire weak solution (u, v) of inequality* (1) *on $\mathbb{R}^n$ such that $u(x) > v(x)$.*

References

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[2] V.V. Kurta, Some problems of qualitative theory for nonlinear second-order equations, Doctoral Dissert., Steklov Math. Inst., Moscow, 1994.