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Algebra

Formulas for the Connes–Moscovici Hopf algebra

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Abstract

We give explicit formulas for the coproduct and the antipode in the Connes–Moscovici Hopf algebra $\mathcal{H}_{\mathrm{CM}}$. To do so, we first restrict ourselves to a sub-Hopf algebra $\mathcal{H}^1$ containing the nontrivial elements, namely those for which the coproduct and the antipode are nontrivial. This algebra $\mathcal{H}^1$ is isomorphic to a sub-Hopf algebra of the classical shuffle Hopf algebra which appears naturally in resummation theory, in the framework of formal and analytic conjugacy of vector fields. Using the very simple structure of the shuffle Hopf algebra, we derive explicit formulas for the coproduct and the antipode in $\mathcal{H}_{\mathrm{CM}}$. ***To cite this article: F. Menous, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***
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Résumé

Formules pour l'algèbre de Hopf de Connes–Moscovici. Nous donnons des formules explicites pour le coproduit et l'antipode dans l'algèbre de Hopf de Connes–Moscovici $\mathcal{H}_{\mathrm{CM}}$. Pour ce faire, on se restreint d'abord à la sous-algèbre de Hopf $\mathcal{H}^1$ contenant les éléments non triviaux, i.e. ceux pour lesquels le coproduit et l'antipode sont non triviaux. Cette algèbre est isomorphe à une sous-algèbre de l'algèbre de Hopf des battages qui apparaît naturellement en théorie de la resommation, dans l'étude de la conjugaison formelle et analytique des champs de vecteurs. En utilisant la structure très simple de l'algèbre de Hopf des battages, on déduit des formules explicites pour le coproduit et l'antipode dans $\mathcal{H}_{\mathrm{CM}}$. ***Pour citer cet article : F. Menous, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***
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1. Introduction

The Connes–Moscovici Hopf algebra $\mathcal{H}_{\mathrm{CM}}$ defined in [5] is the enveloping algebra of the Lie algebra which is the linear span of $Y, X, \delta_n, n \geqslant 1$ with the relations,

$$[X,Y]=X, \quad [Y,\delta_n]=n\delta_n, \quad [\delta_n,\delta_m]=0, \quad [X,\delta_n]=\delta_{n+1} \tag{1}$$

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for all $m, n \geqslant 1$. The coproduct cop in $\mathcal{H}_{\mathrm{CM}}$ is defined by

$$\mathrm{cop}(Y) = Y \otimes 1 + 1 \otimes Y, \quad \mathrm{cop}(X) = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y, \quad \mathrm{cop}(\delta_1) = \delta_1 \otimes 1 + 1 \otimes \delta_1 \tag{2}$$

with $\mathrm{cop}(\delta_n)$ defined by induction using (1). The coproduct of X and Y is given, whereas the coproduct of δ_n is nontrivial. Nonetheless, the algebra generated by $\{\delta_n, n \geqslant 1\}$ is a graded sub-Hopf algebra $\mathcal{H}^1 \subset \mathcal{H}_{\mathrm{CM}}$, called the Faà di Bruno algebra, whose graduation is defined by $\mathrm{gr}(\delta_{n_1} \cdots \delta_{n_s}) = n_1 + \cdots + n_s$.

The Hopf algebra $\mathcal{H}^1$ is strongly linked to the Lie algebra of formal vector fields on the line: let $\mathcal{A}^1$ the Lie algebra of formal vector fields generated by the derivations $\mathbb{B}_n = x^{n+1}\partial_x$ ($n \geqslant 1$). Its enveloping algebra $\mathcal{U}(\mathcal{A}^1)$ is a graded Hopf algebra and, see [5], the Hopf algebra $\mathcal{H}^1$ is the dual of $\mathcal{U}(\mathcal{A}^1)$. Note that this dual is well-defined, since the graded components of $\mathcal{U}(\mathcal{A}^1)$ are finite dimensional vector spaces. If $G(\mathcal{A}^1) \subset \mathcal{U}(\mathcal{A}^1)$ is the group of the group-like elements of $\mathcal{U}(\mathcal{A}^1)$, it can be identified to the group G_2 of formal diffeomorphisms of $\mathbb{R}$, of the form $\psi(x) = x + o(x)$ by the equality

$$\forall \mathbb{F} \in G(\mathcal{A}^1), \quad \mathbb{F} \cdot f = f \circ \psi^{-1}, \quad f \text{ function on } \mathbb{R} \tag{3}$$

and, see [5], if, for $n \geqslant 1$, the functional γ_n on G_2 is defined by

$$\gamma_n(\psi^{-1}) = \left(\partial_x^n \log \psi'(x)\right)_{x=0} \tag{4}$$

then the equality $\Theta(\delta_n) = \gamma_n$ determines a canonical isomorphism Θ of $\mathcal{H}^1$ with the Hopf algebra of coordinates on the group G_2. This does not help much in finding explicit formulas for $\mathcal{H}^1$ but this definitely suggests a link between $\mathcal{H}^1$ and the shuffle Hopf algebra.

2. The shuffle Hopf algebra

The origin of the present work is based on the following remark: the Lie algebra $\mathcal{A}^1$ and the group $G(\mathcal{A}^1) = G_2$ appear naturally in the framework of formal and analytic conjugacy of local analytic vector fields. There has been an extensive work on this problem (see, for example, [1–3]) and, roughly speaking, the formal conjugating diffeomorphism, if it exists, is generally defined with the help of the elements $\mathbb{B}_n$ of $\mathcal{A}^1$, where, in the computations, one can consider that the elements $\mathbb{B}_n$ *freely* generate a free Lie algebra.

These remarks suggest the introduction of, by analogy with $\mathcal{A}^1$, the graded free Lie algebra A^1, generated by a set of primitive elements Δ_n, $n \geqslant 1$,

$$\mathrm{cop}(\Delta_n) = \Delta_n \otimes 1 + 1 \otimes \Delta_n. \tag{5}$$

The enveloping algebra $\mathcal{U}(A^1)$ is a Hopf algebra which is also called the concatenation Hopf algebra in combinatorics (see [4]). If the unity is $\Delta_\emptyset = 1$, then a basis of the vector space $\mathcal{U}(A^1)$ is given by the elements

$$\Delta_{n_1} \cdots \Delta_{n_s} = \Delta_{n_1,\ldots,n_s} \tag{6}$$

where $(n_1, \ldots, n_s)$ is in

$$\mathcal{N} = \left\{\boldsymbol{n} = (n_1, \ldots, n_s) \in (\mathbb{N}^*)^s,\ s \geqslant 0\right\} \tag{7}$$

with $(n_1, \ldots, n_s) = \emptyset$ if $s = 0$.

The structure of the enveloping algebra $\mathcal{U}(A^1)$ can be described as follows: the product is given by

$$\forall \boldsymbol{m}, \boldsymbol{n} \in \mathcal{N}, \quad \Delta_{\boldsymbol{m}} \Delta_{\boldsymbol{n}} = \Delta_{\boldsymbol{mn}} \quad \text{(concatenation)}, \tag{8}$$

the coproduct is

$$\mathrm{cop}(\Delta_{\boldsymbol{n}}) = \sum_{\boldsymbol{n}^1, \boldsymbol{n}^2} \mathrm{sh}_{\boldsymbol{n}}^{\boldsymbol{n}^1, \boldsymbol{n}^2} \Delta_{\boldsymbol{n}^1} \otimes \Delta_{\boldsymbol{n}^2} \tag{9}$$

where $\mathrm{sh}_{\boldsymbol{n}}^{\boldsymbol{n}^1,\boldsymbol{n}^2}$ is the number of shuffling of the sequences $\boldsymbol{n}^1, \boldsymbol{n}^2$ that gives $\boldsymbol{n}$. Finally, the antipode S is defined by

$$S(\Delta_{n_1,\dots,n_s}) = (-1)^s \Delta_{n_s,\dots,n_1}. \tag{10}$$

Once again one can define the group $G(A^1)$. Thanks to the graduation on $\mathcal{U}(A^1)$, its dual H^1 is a Hopf algebra, the Hopf algebra of coordinates on $G(A^1)$ and, if the dual basis of $\{\Delta_{\boldsymbol{n}}, \boldsymbol{n} \in \mathcal{N}\}$ is $\{Z^{\boldsymbol{n}}, \boldsymbol{n} \in \mathcal{N}\}$ then the product in H^1 is defined by:

$$\forall \boldsymbol{n}^1, \boldsymbol{n}^2, \quad Z^{\boldsymbol{n}^1} Z^{\boldsymbol{n}^2} = \sum_{\boldsymbol{n}} \mathrm{sh}_{\boldsymbol{n}}^{\boldsymbol{n}^1,\boldsymbol{n}^2} Z^{\boldsymbol{n}}. \tag{11}$$

The coproduct is:

$$\mathrm{cop}(Z^{\boldsymbol{n}}) = \sum_{\boldsymbol{n}^1\boldsymbol{n}^2=\boldsymbol{n}} Z^{\boldsymbol{n}^1} \otimes Z^{\boldsymbol{n}^2}, \tag{12}$$

where $\boldsymbol{n}^1\boldsymbol{n}^2$ is the concatenation of the two sequences. Finally, the antipode is given by

$$S(Z^{n_1,\dots,n_s}) = (-1)^s Z^{n_s,\dots,n_1}. \tag{13}$$

The structure of H^1 (coproduct, antipode, ...) is fully explicit. This will be of great use since one can define a surjective morphism from A^1 onto $\mathcal{A}^1$ that induces an injective morphism from $\mathcal{H}^1$ into H^1. In other words, $\mathcal{H}^1$ can be identified with a sub-Hopf algebra of H^1 and, as everything is explicit in H^1, one can derive formulas for the coproduct and the antipode in $\mathcal{H}^1$.

3. Morphisms

The map defined by $\rho(\Delta_n) = \mathbb{B}_n = x^{n+1}\partial_x$ obviously determines a surjective morphism from A^1 (resp. $\mathcal{U}(A^1)$, resp. $G(A^1)$) onto $\mathcal{A}^1$ (resp. $\mathcal{U}(\mathcal{A}^1)$, resp. $G(\mathcal{A}^1) = G_2$) since $\{\Delta_n\}_{n\geqslant 1}$ (resp. $\{\mathbb{B}_n\}_{n\geqslant 1}$) is a generating family of primitive elements of $\mathcal{U}(A^1)$ (resp. $\mathcal{U}(\mathcal{A}^1)$). By duality, it induces a morphism ρ^* from $\mathcal{H}^1$ to H^1 by

$$\forall \delta \in \mathcal{H}^1, \quad \rho^*(\delta) = \delta \circ \rho \tag{14}$$

and, since ρ is surjective, ρ^* is injective: $\mathcal{H}^1$ is isomorphic to the sub-Hopf algebra $\rho^*(\mathcal{H}^1) \subset H^1$. Using this injective morphism, we define

$$\forall n \geqslant 1, \quad \Gamma_n = \rho^*(\delta_n) \tag{15}$$

and $\rho^*(\mathcal{H}^1)$ is then the Hopf algebra generated by the Γ_ns. In order to get formulas in $\mathcal{H}^1$, we will use the algebra $\rho^*(\mathcal{H}^1)$ and express the Γ_ns in terms of the $Z^{\boldsymbol{n}}$s:

Theorem 3.1. *For $n \geqslant 1$,*

$$\Gamma_n = n! \sum_{\boldsymbol{n}=(n_1,\dots,n_s)\in\mathcal{N}_n} Q^{\boldsymbol{n}} Z^{\boldsymbol{n}} \tag{16}$$

where $\boldsymbol{n} \in \mathcal{N}_n$ if $n_1 + \cdots + n_s = n$ and $Q^{n_1,\dots,n_s} = (n_s + 1)\prod_{i=2}^{s} \hat{n}_i$ with $Q^{n_1} = (n_1 + 1)$ and $\hat{n}_i = n_i + \cdots + n_s$.

The proof is based on the definition of ρ^* and on the formula for the product in H^1. For example, let $F = \sum_{\boldsymbol{n}\in\mathcal{N}} M^{\boldsymbol{n}} \Delta_{\boldsymbol{n}} \in G(A^1)$ where $M^{\emptyset} = 1$ and $M^{\boldsymbol{n}} = Z^{\boldsymbol{n}}(F) \in \mathbb{R}$ satisfy formula (11). If $\rho(F)$ is identified to the element ψ of G_2 by Eq. (3) then

$$\psi^{-1}(x) = \rho(F)\cdot x = x + M^1 x^2 + (2M^{1,1} + M^2)x^3 + \cdots. \tag{17}$$

From this, one can deduce that $\Gamma_2(F) = \gamma_2(\psi) = 2!(6M^{1,1} + 3M^2 - 2M^1M^1)$ but, since $M^1M^1 = 2M^{1,1}$

$$\Gamma_2(F) = \gamma_2(\psi) = 2!(2M^{1,1} + 3M^2) = 2!(2Z^{1,1} + 3Z^2)(F). \tag{18}$$

4. Formulas

As $\rho^*(\delta_n) = \Gamma_n \in H^1$, and, since the coproduct and the antipode are explicit in H^1, we get, after some heavy recursive computations, the following formulas:

Theorem 4.1. *For* $n \geqslant 1$,

$$\operatorname{cop}(\delta_n) = \delta_n \otimes 1 + 1 \otimes \delta_n + \sum_{(n_1,\dots,n_{s+1}) \in \mathcal{N}_n, s \geqslant 1} \frac{n!}{n_1! \cdots n_s! n_{s+1}!} \alpha_{n_{s+1}}^{n_1,\dots,n_s} (\delta_{n_1} \cdots \delta_{n_s}) \otimes \delta_{n_{s+1}} \tag{19}$$

and, for $\boldsymbol{n} = (n_1, \dots, n_s) \in \mathcal{N}/\{\emptyset\}$ $(l(\boldsymbol{n}) = s)$ *and* $m \geqslant 1$,

$$\alpha_m^{\boldsymbol{n}} = \sum_{t=1}^{l(\boldsymbol{n})} C_m^t \sum_{\boldsymbol{n}^1 \cdots \boldsymbol{n}^t = \boldsymbol{n}, \boldsymbol{n}^i \neq \emptyset} \frac{1}{l(\boldsymbol{n}^1)! \cdots l(\boldsymbol{n}^t)!} \prod_{i=1}^{t} \frac{1}{\|\boldsymbol{n}^i\| + 1} \tag{20}$$

where, for $\boldsymbol{n} = (n_1, \dots, n_s) \in \mathcal{N}$, $l(\boldsymbol{n}) = s$, $\|\boldsymbol{n}\| = n_1 + \cdots + n_s$ *and with the convention* $C_m^t = \frac{m!}{t!(m-t)!} = 0$ *if* $t > m$.

For the antipode S:

Theorem 4.2. *For* $n \geqslant 1$,

$$S(\delta_n) = \sum_{\boldsymbol{n} = (n_1,\dots,n_s) \in \mathcal{N}_n} \frac{n!}{n_1! \cdots n_s!} \beta^{n_1,\dots,n_s} \delta_{n_1} \cdots \delta_{n_s} \tag{21}$$

with $\beta^{n_1} = -1$ *and, if* $(n_1, \dots, n_{s+1}) \in \mathcal{N}$ $(s \geqslant 1)$ *and* $\boldsymbol{n} = (n_1, \dots, n_s)$,

$$\beta^{n_1,\dots,n_s,n_{s+1}} = \sum_{t=1}^{s} \sum_{\boldsymbol{n}^1 \cdots \boldsymbol{n}^t = \boldsymbol{n}, \boldsymbol{n}^i \neq \emptyset} B_{n_{s+1}}^{\|\boldsymbol{n}^1\|,\dots,\|\boldsymbol{n}^t\|} \frac{1}{l(\boldsymbol{n}^1)! \cdots l(\boldsymbol{n}^t)!} \prod_{i=1}^{t} \frac{1}{\|\boldsymbol{n}^i\| + 1} \tag{22}$$

where, if $\boldsymbol{m} = (m_1, \dots, m_t) \in \mathcal{N}/\{\emptyset\}$ *and* $k \geqslant 1$,

$$B_k^{\boldsymbol{m}} = \sum_{i=1}^{l(\boldsymbol{m})} (-1)^{i-1} \sum_{\boldsymbol{m}^1 \cdots \boldsymbol{m}^i = \boldsymbol{m},\ \boldsymbol{m}^j \neq \emptyset} \prod_{j=1}^{i} C_{k + \|\boldsymbol{m}^{j+1}\| + \|\boldsymbol{m}^{j+2}\| + \cdots + \|\boldsymbol{m}^i\|}^{l(\boldsymbol{m}^j)} \tag{23}$$

with $\|\boldsymbol{m}^{j+1}\| + \|\boldsymbol{m}^{j+2}\| + \cdots + \|\boldsymbol{m}^i\| = 0$ *if* $j = i$.

These computations are nontrivial but, at least, we have formulas for the coproduct and the antipode in the Connes–Moscovici Hopf algebra. Note that these formulas are not unique, because $\mathcal{H}^1$ is commutative but, in the computations, it is much more 'simple' to consider that the algebra generated by the δ_n is noncommutative.

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