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Algebraic Geometry

On the derived category of coherent sheaves on a 5-dimensional Fano variety ☆

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Abstract

Let $LG_3^{\mathbf{C}}$ be the Grassmannian of Lagrangian planes in a six-dimensional vector space V. It is a six-dimensional Fano variety of index 4. Consider its smooth hyperplane section. We show that in the derived category of coherent sheaves on such a hyperplane section there exists an exceptional collection, generating the derived category. ***To cite this article: A. Samokhin, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Sur la catégorie dérivée de faisceaux cohérents sur une variété de Fano de dimension 5. Soit $LG_3^{\mathbf{C}}$ la grassmannienne des plans lagrangiens dans un espace vectoriel V de dimension 6. C'est une variété de Fano d'indice 4. Considérons sa section lisse par un hyperplan. Nous montrons que dans la catégorie dérivée des faisceaux cohérents sur une telle section il existe une collection exceptionnelle qui engendre la catégorie dérivée. ***Pour citer cet article : A. Samokhin, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Version française abrégée

L'objet de ce travail est de prouver un théorème sur la catégorie dérivée des faisceaux cohérents sur une variété de Fano de dimension 5. La variété que nous examinerons dans cet article peut être obtenue de la manière suivante : considérons la grassmannienne des plans lagrangiens $LG_3^{\mathbf{C}}$; par définition cette variété est une sous-variété lisse de la grassmannienne $Gr_{3,6}$ et peut être considérée comme le lieu des zéros d'une section générique du fibré vectoriel

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$\bigwedge^2(S^*_{Gr_{3,6}})$, $S_{Gr_{3,6}}$ étant le fibré universel sur $Gr_{3,6}$. La grassmannienne $LG_3^{\mathbf{C}}$ est plongée canoniquement dans $\mathbb{P}^{13}$. Un hyperplan générique par rapport à ce plongement coupe $LG_3^{\mathbf{C}}$ le long d'une variété lisse et irréductible X. On demontre que les nombres de Hodge de X sont égaux à ceux d'un espace projectif de dimension 5. Des variétés de Fano de dimension n ayant les nombres de Hodge égaux à ceux de $\mathbb{P}^n$ sont dites minimales. L'intérêt de telles variétés du point de vue des catégories dérivées des faisceaux cohérents découle de la conjecture suivante :

Conjecture (A. Bondal). *La catégorie dérivée des faisceaux cohérents d'une variété Fano minimale est engendrée par une collection exceptionnelle.*

Rappelons qu'une collection d'objets $(E_0, E_1, \ldots, E_n)$ dans une catégorie triangulée k-linéaire, k étant un corps de base algébriquement clos de caractéristique zero, est dite exceptionnelle si les deux propriétés suivantes sont satisfaites :

(i) Chaque objet E_i, $i = 0, 1, \ldots, n$, est exceptionnel, i.e. $\mathrm{R}\mathit{Hom}_D(E_i, E_i) = k$ comme k-algèbre ;
(ii) La collection ordonnée des objets $(E_0, E_1, \ldots, E_n)$ est semiorthogonale, i.e. $\mathrm{R}\mathit{Hom}_D(E_i, E_j) = 0$ pour $i > j$.

On dit que la catégorie D est engendrée par une collection exceptionnelle si la plus petite sous-catégorie pleine triangulée de D, qui contient la collection donnée, coïncide avec D.

La conjecture au-dessus est inspirée par la symétrie miroir homologique [5].

Cette conjecture est vraie jusqu'en dimension 3 (voir [1,4,8,6]). Dans cette note on prouve le théorème suivant :

Théorème 0.1. *La catégorie dérivée $\mathcal{D}^b(\mathit{Coh}\, X)$ est engendrée par une collection exceptionnelle.*

La preuve de ce théorème s'appuie sur le résultat principal de [9], notamment sur le fait qu'il existe une collection exceptionnelle complète dans la catégorie dérivée $\mathcal{D}^b(\mathit{Coh}\, LG_3^{\mathbf{C}})$. On exhibe d'abord une collection exceptionnelle dans $\mathcal{D}^b(\mathit{Coh}\, X)$ qui engendre le groupe de Grothendieck $K^0(X)$. Soit L cette collection. Il convient de démontrer (voir, e.g. [2]) que l'orthogonal droit à la sous-catégorie $\langle L\rangle$ engendrée par la collection L est zéro. Supposons qu'il existe un objet non nul de l'orthogonal droite $\langle L\rangle^\perp$. On montre que pour un objet de $\langle L\rangle^\perp$ ses faisceaux de cohomologie sont aussi dans $\langle L\rangle^\perp$. Cela permet de conclure que $\langle L\rangle^\perp$ est zéro, d'où le théorème.

1. Introduction

We are working over an algebraically closed field k. Consider the Grassmannian $Gr_{3,6}$ of three-dimensional subspaces of a six-dimensional vector space V. Let $S_{Gr_{3,6}}$ be the universal bundle over $Gr_{3,6}$. Fix a non-degenerate skew-symmetric form ω on the vector space V, $\omega \in \bigwedge^2 V^*$. Such a form gives rise to a section of the bundle $\bigwedge^2 S^*_{Gr_{3,6}}$. Its zero locus consists of isotropic subspaces of maximal dimension in V with respect to the form ω. This is a smooth Fano variety of dimension 6, which is denoted $LG_3^{\mathbf{C}}$. In [9] the following theorem was proved:

Theorem 1.1. *The bounded derived category of coherent sheaves over $LG_3^{\mathbf{C}}$ is generated by an exceptional collection.*

In what follows, given a smooth variety S, its bounded derived category of coherent sheaves is denoted $\mathcal{D}^b(S)$.

Assume given a triangulated category $\mathcal{D}$.

Definition 1.2. A triangulated subcategory $\mathcal{D}_1$ of the category $\mathcal{D}$ is called admissible if the embedding functor $i : \mathcal{D}_1 \hookrightarrow \mathcal{D}$ has left and right adjoints.

Assume given an admissible subcategory $\mathcal{D}_1$ of $\mathcal{D}$.

Definition 1.3. The full subcategory of $\mathcal{D}$ generated by the objects $E \in \mathcal{D}$ such that $Hom_{\mathcal{D}}(F, E) = 0$ for any $F \in \mathcal{D}_1$ is called right orthogonal to $\mathcal{D}_1$ and is denoted $\mathcal{D}_1^{\perp}$. The left orthogonal is defined similarly.

If $\mathcal{D}_1$ is admissible then it follows that $\mathcal{D}$ has a semiorthogonal decomposition: $\mathcal{D} = \langle \mathcal{D}_1^{\perp}, \mathcal{D}_1 \rangle$ (see [2]).

Definition 1.4. An exceptional collection in the triangulated category D is said to generate D if the smallest full triangulated subcategory of D containing this collection coincides with the whole D.

Assume given an exceptional collection of objects in $\mathcal{D}$. It was proved in [3] that a triangulated subcategory generated by an exceptional collection in $\mathcal{D}$ is admissible. In particular, the fact that an exceptional collection generates $\mathcal{D}$ amounts to saying that the right orthogonal to the subcategory generated by this collection is zero. The usefulness of such exceptional collections (complete exceptional collections for short) for the study of derived categories comes from the fact that such a collection gives rise to a new (non-standard) t-structure in the ambient derived category (see [2]). The collection mentioned in Theorem 1.1 consists of eight locally free sheaves: $S_{LG_3^{\mathbf{C}}}(-3)$, $\mathcal{O}_{LG_3^{\mathbf{C}}}(-3)$, $S_{LG_3^{\mathbf{C}}}(-2)$, $\mathcal{O}_{LG_3^{\mathbf{C}}}(-2)$, $S_{LG_3^{\mathbf{C}}}(-1)$, $\mathcal{O}_{LG_3^{\mathbf{C}}}(-1)$, $S_{LG_3^{\mathbf{C}}}$, $\mathcal{O}_{LG_3^{\mathbf{C}}}$.

The aim of the present Note is to prove a result which is similar to Theorem 1.1. Namely, we start again with the variety $LG_3^{\mathbf{C}}$. It is canonically embedded into $\mathbb{P}^{13}$. Take a smooth hyperplane section of $LG_3^{\mathbf{C}}$ with respect to this embedding. Such a hyperplane section is a five-dimensional Fano variety. Denote it by X. By adjunction one sees that X is a Fano variety of index 3 [9].[1] We are going to prove the following:

Theorem 1.5. *The derived category $\mathcal{D}^b(X)$ is generated by an exceptional collection.*

We prove this in the next section.

2. Proof of Theorem 1.5

We start with two preliminary lemmas.

Lemma 2.1. *Hodge numbers of X are given by $h_X^{i,j} = 0$, $i \neq j$; $h_X^{i,i} = 1$, $i = 0, 1, \ldots, 5$.*

Thus, Hodge numbers of X are the same as those of $\mathbb{P}^5$. Lemma 2.1 implies that $Pic_X = \mathbb{Z}$. Let $\mathcal{O}_X(1)$ be the positive generator of this group. The restriction to X of the universal bundle $S_{LG_3^{\mathbf{C}}}$ is denoted by S_X.

Lemma 2.2. *The collection $L = (S_X(-2), \mathcal{O}_X(-2), S_X(-1), \mathcal{O}_X(-1), S_X, \mathcal{O}_X)$ is an exceptional collection in $\mathcal{D}^b(X)$.*

Lemma 2.1 also entails that $rk\, K^0(X) = 6$. It follows from Lemmas 2.1 and 2.2 that the collection L is a semi-orthogonal basis in the Grothendieck group $K^0(X)$, where semiorthogonality is understood in the sense that the Gram matrix of the Euler form $\chi([A],[B]) = \sum_i (-1)^i \dim Ext^i([A],[B])$ on $K^0(X)$, $[A],[B]$ being two classes in $K^0(X)$, is uppertriangular with units along the diagonal. Hence, it would be reasonable to conjecture that the collection L generates $\mathcal{D}^b(X)$.

[1] Let Z be a Fano variety. Recall that the index $i(Z)$ of Z is the largest number r such that there exists a divisor H with $\omega_Z = rH$, ω_Z being the canonical class of Z.

Theorem 2.3. *The collection L generates $\mathcal{D}^b(X)$.*

Proof. Let i be the embedding $X \overset{i}{\hookrightarrow} LG_3^{\mathbf{C}}$. Let $\langle L\rangle$ be the smallest triangulated subcategory of $\mathcal{D}^b(X)$ generated by L. Suppose that L is not complete. Then, by [2], it follows that in this case the right orthogonal $\langle L\rangle^{\perp}$ is non-zero. Let $\mathcal{F}$ be a non-zero object in $\langle L\rangle^{\perp}$. Take the push-forward $i_*(\mathcal{F})$. Note that i_* is an exact functor since it is the direct image functor with respect to a closed embedding. Let $\mathcal{E}$ be any object in $\mathcal{D}^b(LG_3^{\mathbf{C}})$. Then the adjunction of push-forward and pull-back functors implies the equality: $\mathrm{R}\mathit{Hom}_{LG_3^{\mathbf{C}}}(i^*\mathcal{E},\mathcal{F}) = \mathrm{R}\mathit{Hom}_X(\mathcal{E}, i_*\mathcal{F})$. If one takes objects $S_{LG_3^{\mathbf{C}}}$, $\mathcal{O}_{LG_3^{\mathbf{C}}}$, as well as their twists by $\mathcal{O}_{LG_3^{\mathbf{C}}}(-1)$ and $\mathcal{O}_{LG_3^{\mathbf{C}}}(-2)$, and plugs them into this formula then one sees[2] that the object $i_*\mathcal{F}$ is right orthogonal in $\mathcal{D}^b(LG_3^{\mathbf{C}})$ to the following set of sheaves: $S_{LG_3^{\mathbf{C}}}(-2)$, $\mathcal{O}_{LG_3^{\mathbf{C}}}(-2)$, $S_{LG_3^{\mathbf{C}}}(-1)$, $\mathcal{O}_{LG_3^{\mathbf{C}}}(-1)$, $S_{LG_3^{\mathbf{C}}}$, $\mathcal{O}_{LG_3^{\mathbf{C}}}$.

A corollary to Theorem 1.1 says that this set of sheaves is an exceptional collection and that it generates a full triangulated subcategory in $\mathcal{D}^b(X)$. Moreover, the right orthogonal to the subcategory generated by this collection can be identified with a full triangulated subcategory generated by the exceptional pair $S_{LG_3^{\mathbf{C}}}(-3), \mathcal{O}_{LG_3^{\mathbf{C}}}(-3)$. By [2], the object $i_*(\mathcal{F})$ can be included in a distinguished triangle

$$\cdots \longrightarrow \mathcal{O}_{LG_3^{\mathbf{C}}}(-3)\otimes V^{\cdot} \longrightarrow i_*\mathcal{F} \longrightarrow S_{LG_3^{\mathbf{C}}}(-3)\otimes W^{\cdot} \xrightarrow{[1]} \cdots \tag{1}$$

where $V^{\cdot}$ and $W^{\cdot}$ are graded vector spaces and we write $E\otimes V^{\cdot}$ for $\bigoplus_i E\otimes V^i[-i]$, the direct sum being a complex with zero differential. The triangle (1) is a distinguished triangle in $\mathcal{D}^b(LG_3^{\mathbf{C}})$, hence one can apply to it the cohomological functor $\mathcal{H}^0 = \tau_{\leqslant 0}\tau_{\geqslant 0}$[3], producing a long exact sequence of sheaves on $LG_3^{\mathbf{C}}$:

$$\cdots \longrightarrow S_{LG_3^{\mathbf{C}}}(-3)\otimes W^1 \longrightarrow \mathcal{O}_{LG_3^{\mathbf{C}}}(-3)\otimes V^0 \longrightarrow \mathcal{H}^0(i_*\mathcal{F}) \longrightarrow S_{LG_3^{\mathbf{C}}}(-3)\otimes W^0 \longrightarrow \cdots. \tag{2}$$

Since the push-forward functor is exact, it commutes with taking cohomologies: $\mathcal{H}^k(i_*\mathcal{F}) = i_*\mathcal{H}^k(\mathcal{F})$. Hence, the exact sequence (2) can be rewritten as follows:

$$\cdots \longrightarrow S_{LG_3^{\mathbf{C}}}(-3)\otimes W^1 \longrightarrow \mathcal{O}_{LG_3^{\mathbf{C}}}(-3)\otimes V^0 \longrightarrow i_*(\mathcal{H}^0\mathcal{F}) \longrightarrow S_{LG_3^{\mathbf{C}}}(-3)\otimes W^0 \longrightarrow \cdots. \tag{3}$$

All maps in this long exact sequence are morphisms of coherent sheaves on $LG_3^{\mathbf{C}}$. Consider a map $i_*\mathcal{H}^k(F) \to S_{LG_3^{\mathbf{C}}}(-3)\otimes W^k$. Then either the latter sheaf is zero, thus $W^k = 0$, or there is a map between a sheaf supported in X and a locally free sheaf over $LG_3^{\mathbf{C}}$. However, the former sheaf, being a torsion sheaf, has no non-zero maps into a locally free one. Thus, all maps $i_*\mathcal{H}^k(F) \to S_{LG_3^{\mathbf{C}}}(-3)\otimes W^k$ are zero. In other words, the long exact sequence (2) is split up into the direct sum of short exact sequences:

$$\bigoplus_k \big[0 \longrightarrow S_{LG_3^{\mathbf{C}}}(-3)\otimes W^{k+1} \longrightarrow \mathcal{O}_{LG_3^{\mathbf{C}}}(-3)\otimes V^k \longrightarrow i_*(\mathcal{H}^k\mathcal{F}) \longrightarrow 0\big], \tag{4}$$

each short exact sequence being shifted appropriately in the derived category. Consider any direct summand of this sum:

$$E^k : 0 \longrightarrow S_{LG_3^{\mathbf{C}}}(-3)\otimes W^{k+1} \longrightarrow \mathcal{O}_{LG_3^{\mathbf{C}}}(-3)\otimes V^k \longrightarrow i_*(\mathcal{H}^k\mathcal{F}) \longrightarrow 0. \tag{5}$$

E^k is an acyclic complex in $\mathcal{D}^b(LG_3^{\mathbf{C}})$. Applying to this complex the functor $\mathrm{R}\mathit{Hom}_{LG_3^{\mathbf{C}}}(?, E^k)$, where ? runs over the set of objects $S_{LG_3^{\mathbf{C}}}(-2)$, $\mathcal{O}_{LG_3^{\mathbf{C}}}(-2)$, $S_{LG_3^{\mathbf{C}}}(-1)$, $\mathcal{O}_{LG_3^{\mathbf{C}}}(-1)$, $S_{LG_3^{\mathbf{C}}}$, $\mathcal{O}_{LG_3^{\mathbf{C}}}$, we get long exact cohomology sequences:

$$\cdots \to \mathit{Ext}^j_{LG_3^{\mathbf{C}}}\big(?, S_{LG_3^{\mathbf{C}}}(-3)\otimes W^{k+1}\big) \to \mathit{Ext}^j_{LG_3^{\mathbf{C}}}\big(?, \mathcal{O}_{LG_3^{\mathbf{C}}}(-3)\otimes V^k\big) \to \mathit{Ext}^j_{LG_3^{\mathbf{C}}}\big(?, i_*(\mathcal{H}^k\mathcal{F})\big) \to \cdots. \tag{6}$$

[2] Recall that by definition $S_X := i^*(S_{LG_3^{\mathbf{C}}})$.

[3] Truncation functors $\tau_{\leqslant 0}, \tau_{\geqslant 0}$ are taken with respect to the standard t-structure in $\mathcal{D}^b(\mathit{Coh}\,LG_3^{\mathbf{C}})$.

First two *Ext* groups in this long exact sequence are equal to zero for any j since the collection $S_{LG_3^{\mathbb{C}}}(-3)$, $\mathcal{O}_{LG_3^{\mathbb{C}}}(-3)$, $S_{LG_3^{\mathbb{C}}}(-2)$, $\mathcal{O}_{LG_3^{\mathbb{C}}}(-2)$, $S_{LG_3^{\mathbb{C}}}(-1)$, $\mathcal{O}_{LG_3^{\mathbb{C}}}(-1)$, $S_{LG_3^{\mathbb{C}}}$, $\mathcal{O}_{LG_3^{\mathbb{C}}}$ is exceptional. Hence $Ext^j_{LG_3^{\mathbb{C}}}(?, i_*(\mathcal{H}^k\mathcal{F}))$ $= 0$ for any j. Again by adjunction one gets $Ext^j_X(i^*(?), \mathcal{H}^k\mathcal{F}) = 0$. This implies that $\mathcal{H}^k(\mathcal{F})$ belongs to $\langle L\rangle^{\perp}$. Recall that according to the remark after Lemma 2.2, the collection $\langle L\rangle$ generates the group $K^0(X)$. Hence any object in $\langle L\rangle^{\perp}$ represents a zero class in $K^0(X)$. Thus, the sheaf $\mathcal{H}^k(\mathcal{F})$ is equal to zero in $K^0(X)$. It follows immediately that $\mathcal{H}^k(\mathcal{F})$ is zero itself. The object $\mathcal{F}$, however, having zero cohomology, is quasiisomorphic to zero, hence the statement of Theorem 1.5. □

Remark 1. The main result of this paper should, by another method, follow from more general results by Kuznetsov [7].

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