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Mathematical Analysis

Piatetski-Shapiro phenomenon in the uniqueness problem

Nir Lev, Alexander Olevskii

School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel

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Presented by Jean-Pierre Kahane

Abstract

We extend the phenomenon discovered by Piatetski-Shapiro (1954) to l^q spaces. To be precise, for any $q > 2$ we construct a compact K on the circle, which supports a distribution S with Fourier transform $\widehat{S} \in l^q$, but does not support such a measure. ***To cite this article: N. Lev, A. Olevskii, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Phénomène de Piatetski-Shapiro et le problème d'unicité. Nous étendons aux espaces l^q le phénomène découvert par Piatetski-Shapiro en 1954 : pour tout $q > 2$ nous construisons un compact K sur le cercle, qui porte une distribution dont la transformée de Fourier appartient à l^q, mais qui ne porte pas de mesure ayant cette propriété. ***Pour citer cet article : N. Lev, A. Olevskii, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

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L'origine de ce travail se trouve dans les théorèmes de Cantor sur l'unicité du développement trigonométrique et dans la découverte par Menchoff d'une série trigonométrique non nulle qui converge vers zéro hors d'un ensemble fermé F de mesure de Lebesgue nulle.

La série produite par Menchoff était une série de Fourier–Stieltjes. En 1954, Piatetski-Shapiro a mis en évidence un fermé F de multiplicité, c'est-à-dire ayant la propriété de Menchoff, mais tel que n'existe aucune série de Fourier–Stieltjes non nulle qui converge vers zéro hors de F. Si l'on considère F comme une partie du cercle, le « phénomène de Piatetski-Shapiro » revient à dire que F peut porter une distribution dont la transformée de Fourier appartient à c_0 mais aucune mesure ayant cette propriété.

E-mail addresses: levnir@post.tau.ac.il (N. Lev), olevskii@post.tau.ac.il (A. Olevskii).

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Cette Note montre que ce phénomène a lieu en remplaçant c_0 par l^q avec $q > 2$ donné. La construction de F s'inspire d'une construction de Kaufman et Körner qui permet de retrouver le phénomène pour c_0. Elle exige une série d'étapes qui sont détaillées sous la forme de lemmes. Le Lemme 2.5 établit la non-existence d'une mesure, et les Lemmes 3.1 et 3.2 l'existence d'une distribution ayant la propriété voulue.

1. Introduction

Piatetski-Shapiro in [8] constructed a compact K on the circle group $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, which supports a (non-zero) Schwartz distribution S with Fourier coefficients $\widehat{S}(n)$ tending to zero ($|n| \to \infty$), but does not support such a measure. This result answered a problem in uniqueness theory of trigonometric expansions, and it was the subject of further development by Körner [5,6], Kaufman [3] and other researchers (see [1,2,4] for more details).

It is known from potential theory that no such a phenomenon can exist in certain weighted l^2 spaces. Namely, if a compact K supports a distribution S such that $\sum_{n\neq 0} |\widehat{S}(n)|^2 |n|^{\alpha-1} < \infty$ (for some $0 \leqslant \alpha < 1$) then it also supports a positive measure satisfying this property (see [2, Chapter III]).

We are interested in the following question: what can be said about l^q spaces? Clearly only the case $q > 2$ is non-trivial. We prove the following:

Theorem 1.1. *For any $q > 2$ there is a compact K on the circle, which supports a (non-zero) distribution S with $\widehat{S} \in l^q$, but does not support such a measure.*

Our approach is inspired by Kahane's presentation of the Piatetski–Körner–Kaufman's results, see [2, pp. 213–216]. The main ingredients of our construction are Riesz products and probabilistic exponential estimates.

2. Lemmas

Below $q > 2$ is a fixed number, $p = q/(q-1)$. We denote by A_q the Banach space of Schwartz distributions S on $\mathbb{T}$, satisfying the condition

$$\|S\|_{A_q} := \|\widehat{S}\|_{l^q} < \infty,$$

and by $M(K)$ the space of finite (complex) Borel measures supported by K, with the usual norm.

We start with the classical exponential estimate:

Lemma 2.1 (S.N. Bernstein). *Let $g_1, \dots, g_N$ be independent random variables on a probability space (Ω, P). Denote*

$$Y = \sum_{j=1}^{N} g_j, \qquad d = \sup_{1\leqslant j\leqslant N} \|g_j\|_\infty.$$

Then, for any $\alpha > 0$,

$$P\bigl\{|Y - \mathbb{E}Y| > \alpha\bigr\} < 2\exp\left(-\frac{\alpha^2}{8Nd^2}\right). \tag{1}$$

The constant $\frac{1}{8}$ can be improved (see for example [7, Chapter III]), but this is not important for us.

Lemma 2.2 (Kahane, see [2, p. 214]). *Given $\delta > 0$, one can find a measure ρ with finite support belonging to the segment $(\frac{1}{3}, \frac{1}{2})$, such that*

$$\int \mathrm{d}\rho = 1 \quad \textit{and} \quad \left|\int s^k \,\mathrm{d}\rho(s)\right| < \delta \quad \textit{for } k = 1, 2, \ldots.$$

Lemma 2.3. *Let g be a 2π-periodic function,*

$$\int_0^{2\pi} g(t)\,\mathrm{d}t = 0, \quad -1 \leqslant g(t) \leqslant 1, \tag{2}$$

which is constant on each interval $(\frac{2\pi(k-1)}{\nu}, \frac{2\pi k}{\nu})$, $1 \leqslant k \leqslant \nu$. Then the system $\{g(\nu^j t)\}_{j=1}^N$ is stochastically independent on the circle $\mathbb{T}$, with respect to the probability measure P defined as

$$\mathrm{d}P(t) = \prod_{j=1}^N \bigl(1 + r_j g(\nu^j t)\bigr)\frac{\mathrm{d}t}{2\pi}, \quad -1 < r_j < 1.$$

This can be checked directly.

Lemma 2.4. *Let $\mu \in A_q$ be a measure supported by a compact K. Then the measure $|\mu|$ belongs to the closure of $A_q \cap M(K)$ in the $M(K)$ norm.*

Proof. There exists a Borel function $\phi : \mathbb{T} \to \mathbb{C}$ with $|\phi| = 1$, such that $\mathrm{d}|\mu| = \phi\,\mathrm{d}\mu$. Given $\varepsilon > 0$, let ψ be a trigonometric polynomial such that $|\mu|\{t \in \mathbb{T}\colon |\phi(t) - \psi(t)| > \varepsilon\} < \varepsilon$ and $|\psi| \leqslant 2$. Then the measure $\mathrm{d}\tilde{\mu} = \psi\,\mathrm{d}\mu$ belongs to $A_q \cap M(K)$ and

$$\bigl\||\mu| - \tilde{\mu}\bigr\|_{M(K)} = \int |\phi - \psi|\,\mathrm{d}|\mu| \leqslant 3\varepsilon + \varepsilon\|\mu\|_{M(K)}. \qquad \square$$

Lemma 2.5. *Suppose a compact K on the circle satisfies the following condition: for any positive integer ν there is a real trigonometric polynomial*

$$X(t) = \sum_{|n| \geqslant \nu} \widehat{X}(n)\,\mathrm{e}^{\mathrm{i}nt}, \quad \|\widehat{X}\|_p \leqslant 1, \quad \frac{1}{100} \leqslant X(t) \leqslant 100 \textit{ on } K.$$

Then K does not support a non-zero measure $\mu \in A_q$.

Proof. Suppose that $\mu \in A_q \cap M(K)$. Given $\varepsilon > 0$, by Lemma 2.4 one can find a measure $\tilde{\mu} \in A_q \cap M(K)$ such that

$$\bigl\||\mu| - \tilde{\mu}\bigr\|_{M(K)} < \varepsilon.$$

Then

$$\left|\int_{\mathbb{T}} X\,\mathrm{d}\tilde{\mu}\right| = \left|\int_K X\,\mathrm{d}\tilde{\mu}\right| \geqslant \int_K X\,\mathrm{d}|\mu| - 100\varepsilon \geqslant \frac{1}{100}\|\mu\| - 100\varepsilon.$$

However, the left-hand side is equal to

$$\left|\sum \widehat{X}(n)\hat{\tilde{\mu}}(-n)\right| \leqslant \biggl(\sum_{|n| \geqslant \nu} \bigl|\hat{\tilde{\mu}}(n)\bigr|^q\biggr)^{1/q} < \varepsilon$$

for sufficiently large ν. Hence $\mu = 0$. $\square$

3. Riesz products and exponential estimates

3.1. Given N, consider the polynomial

$$X(t)=N^{-1/p}\sum_{j=1}^{N}\cos\nu^{j}t \tag{3}$$

where the number $\nu=\nu(N)$ is to be chosen. Set

$$K=K(N)=\left\{t\in\mathbb{T}\colon\ \tfrac{1}{100}\leqslant X(t)\leqslant 100\right\}. \tag{4}$$

Define a Riesz product

$$\lambda_s(t)=\lambda_{N,s}(t)=\prod_{j=1}^{N}\left(1+2sN^{-1/q}\cos\nu^{j}t\right)$$

where the parameter $s\in(\frac{1}{3},\frac{1}{2})$, and $\nu\geqslant 3$. Consider the circle $\mathbb{T}$ as a probability space with the measure $\mathrm{d}\Lambda_s(t)=\lambda_s(t)\frac{\mathrm{d}t}{2\pi}$. The members of the polynomial X are 'almost independent' with respect to this measure. More precisely, given $\delta>0$, approximate the function $\cos t$ with error $<\delta$ (uniformly with respect to t) by a function g constant on each segment $(\frac{2\pi(k-1)}{\nu},\frac{2\pi k}{\nu})$ and satisfying (2). Now use Lemma 2.3 with $r_j=2sN^{-1/q}$, and apply the estimate (1) for the variables $g_j(t)=N^{-1/p}g(\nu^{j}t)$. This gives

$$P\left\{|Y-\mathbb{E}Y|>\alpha\right\}<2\exp\left(-\frac{1}{8}\alpha^{2}N^{2/p-1}\right). \tag{5}$$

Clearly, if $\delta=\delta(N)$ is sufficiently small, this estimate can be extended to the polynomial X on the probability space $(\mathbb{T},\Lambda_s)$. An easy calculation shows $\mathbb{E}(X)=s$, so we get

$$\int\limits_{\{t:\ |X(t)-s|>\alpha\}}\lambda_s(t)\frac{\mathrm{d}t}{2\pi}<2\exp\left(-\frac{1}{8}\alpha^{2}N^{2/p-1}\right), \tag{6}$$

provided that the number $\nu=\nu(N)$ is fixed sufficiently large.

3.2.

Lemma 3.1. *Denote*

$$K'=K'(N)=\left\{t\in\mathbb{T}\colon\ \tfrac{1}{90}\leqslant X(t)\leqslant 90\right\}. \tag{7}$$

Then, given $\delta>0$, *one has for* $N\geqslant N(\delta)$

$$\int\limits_{\mathbb{T}\setminus K'}\lambda_s^{2}(t)\frac{\mathrm{d}t}{2\pi}<\delta.$$

Proof. We have

$$\lambda_s(t)=\prod_{j=1}^{N}\left(1+2sN^{-1/q}\cos\nu^{j}t\right)\leqslant\exp\left(2sN^{-1/q}\sum_{j=1}^{N}\cos\nu^{j}t\right),$$

so using (3) we obtain an estimate

$$\lambda_s(t)\leqslant\exp\left(2sN^{2/p-1}X(t)\right). \tag{8}$$

Combining (6) and (8), it follows that for any $s \in \left(\frac{1}{3}, \frac{1}{2}\right)$,

$$\int\limits_{\{t:\ X(t)<1/90\}} \lambda_s^2(t)\frac{\mathrm{d}t}{2\pi} \leqslant \int\limits_{\{t:\ X(t)<1/90\}} \lambda_s(t)\frac{\mathrm{d}t}{2\pi} \cdot \max_{\{t:\ X(t)<1/90\}} \lambda_s(t)$$
$$< 2\exp\left(-\frac{1}{8}\left(s-\frac{1}{90}\right)^2 N^{2/p-1}\right)\cdot \exp\left(2sN^{2/p-1}\cdot\frac{1}{90}\right) < 2\exp\left(-10^{-3}N^{2/p-1}\right),$$

and, for any integer $k \geqslant 90$,

$$\int\limits_{\{t:\ k<X(t)\leqslant k+1\}} \lambda_s^2(t)\frac{\mathrm{d}t}{2\pi} \leqslant \int\limits_{\{t:\ k<X(t)\leqslant k+1\}} \lambda_s(t)\frac{\mathrm{d}t}{2\pi} \cdot \max_{\{t:\ k<X(t)\leqslant k+1\}} \lambda_s(t)$$
$$< 2\exp\left(-\frac{1}{8}(k-s)^2 N^{2/p-1}\right)\cdot \exp\left(2sN^{2/p-1}(k+1)\right) < 2\exp\left(-N^{2/p-1}\right).$$

Hence, keeping in mind that $X(t) \leqslant N^{1/q}$ for every t, we obtain

$$\int\limits_{\mathbb{T}\setminus K'} \lambda_s^2(t)\frac{\mathrm{d}t}{2\pi} < 2\exp\left(-10^{-3}N^{2/p-1}\right) + 2N^{1/q}\exp\left(-N^{2/p-1}\right) \underset{N\to\infty}{\longrightarrow} 0$$

uniformly in s. □

3.3.

Lemma 3.2. *For any $\varepsilon > 0$, there exists a smooth function f and an integer N such that*
(i) *f is supported by $K(N)$.*
(ii) $f(t) = 1 + \sum_{n\neq 0} \hat{f}(n)\,\mathrm{e}^{\mathrm{i}nt}$, *where* $\sum_{n\neq 0} |\hat{f}(n)|^q < \varepsilon^q$.

Proof. The Fourier coefficients of λ_s which are non-zero can be written

$$\hat{\lambda}_s\left(\sum \tau_j \nu^j\right) = \left(sN^{-1/q}\right)^{\sum|\tau_j|}, \quad \bar{\tau} = (\tau_1, \ldots, \tau_N) \in \{-1, 0, 1\}^N.$$

Given $\delta > 0$, let ρ be the measure from Lemma 2.2,

$$\int \mathrm{d}\rho = 1 \quad \text{and} \quad \left|\int s^k\,\mathrm{d}\rho(s)\right| < \delta \quad (k = 1, 2, \ldots).$$

Define

$$\lambda(t) = \int \lambda_s(t)\,\mathrm{d}\rho(s),$$

then

$$\lambda(t) = \sum_{\bar{\tau}\in\{-1,0,1\}^N} \left(N^{-\frac{1}{q}\sum|\tau_j|}\int s^{\sum|\tau_j|}\,\mathrm{d}\rho(s)\right)\mathrm{e}^{\mathrm{i}(\sum\tau_j\nu^j)t}.$$

It follows that $\hat{\lambda}(0) = 1$ and

$$\sum_{n\neq 0}\left|\hat{\lambda}(n)\right|^q < \delta^q \sum_{\bar{\tau}\in\{-1,0,1\}^N} N^{-\sum|\tau_j|} = \delta^q\left(1+\frac{2}{N}\right)^N < \mathrm{e}^2\delta^q.$$

Now use Lemma 3.1 and choose N such that

$$\left(\int_{\mathbb{T}\setminus K'} \lambda_s^2(t)\frac{\mathrm{d}t}{2\pi}\right)^{1/2} < \frac{\delta}{\|\rho\|_M}.$$

Let $h = \lambda \cdot \mathbf{1}_{K'}$, then this implies

$$\|\lambda - h\|_{A_q} \leqslant \|\lambda - h\|_{L^2(\mathbb{T})} = \|\lambda\|_{L^2(\mathbb{T}\setminus K')} \leqslant \int \|\lambda_s\|_{L^2(\mathbb{T}\setminus K')}\,\mathrm{d}|\rho|(s) < \delta.$$

Clearly if δ is chosen sufficiently small, making a normalization $g = h/\hat{h}(0)$ yields

$$\hat{g}(0) = 1 \quad \text{and} \quad \sum_{n\neq 0}\left|\hat{g}(n)\right|^q < \varepsilon^q.$$

To finish the proof of Lemma 3.2, we define f as the convolution of g with a smooth non-negative function, with integral equals to 1, and which is supported on a sufficiently small neighborhood of 0 to ensure that supp $f \subset K(N)$, which is possible according to (4) and (7). □

3.4. For a sequence $\{\varepsilon_j\}$ define f_j, N_j and $K_j = K(N_j)$ according to Lemma 3.2. We choose ε_j by induction to satisfy the conditions

$$\varepsilon_1 < 2^{-3} \quad \text{and} \quad \|f_1 \cdot f_2 \cdots f_j\|_A \cdot \varepsilon_{j+1} < 2^{-j-3} \quad (j = 1, 2, \ldots)$$

(here A is the Wiener algebra $= A_1$). As in [2, p. 215], this implies that the product $\prod_{j=1}^{\infty} f_j$ will converge in the A_q norm to a non-zero distribution S. Denote $K = \bigcap_{j=1}^{\infty} K_j$, then clearly S is supported on K. On the other hand, due to (3) and (4), Lemma 2.5 ensures that K does not support a non-zero measure in A_q, so Theorem 1.1 is proved.

4. Remarks

The main property of our compact K could be reformulated in spirit of the original Piatetski-Shapiro result, which has distinguished sets of uniqueness and strict uniqueness. Namely:

For any $q > 2$ there is a compact K such that

(i) *There is a non-trivial trigonometric series with coefficients in l^q, converging to zero everywhere outside K.*

(ii) *No Fourier–Stieltjes series may satisfy this property.*

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