Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
C. R. Acad. Sci. Paris, Ser. I 340 (2005) 769–774
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Numerical Analysis

An improved On-Surface Radiation Condition for acoustic scattering problems in the high-frequency spectrum

Xavier Antoine [a,b], Marion Darbas [c], Ya Yan Lu [d]

[a] *Laboratoire de mathématiques pour l'industrie et la physique, CNRS UMR 5640, UFR MIG, 118, route de Narbonne, 31062 Toulouse cedex 4, France*
[b] *Applied and Computational Mathematics, California Institute of Technology, 1220 E. California Boulevard, 217-50, Pasadena, CA 91125, USA*
[c] *Département de génie mathématique, INSA, MIP UMR-5640, complexe scientifique de Rangueil, 31077 Toulouse cedex 4, France*
[d] *Department of Mathematics, City University of Hong-Kong, Tat Chee Avenue, Kowloon, Hong-Kong*

Received 4 April 2005; accepted 8 April 2005

Presented by Philippe G. Ciarlet

Abstract

This Note addresses the derivation of an improved On-Surface Radiation Condition for the numerical solution of the exterior Helmholtz equation at high-frequencies. This condition is built as an approximation of the Neumann-to-Dirichlet map by using a local regularization of its principal classical symbol in the gliding zone for modelling the creeping waves. The numerical simulation of this pseudodifferential operator is efficiently realized with a linear cost according to the dimension of the boundary element approximation space using suitable complex Padé approximants. A numerical example is provided. ***To cite this article: X. Antoine et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Une condition améliorée de radiation de surface pour la diffraction acoustique à haute fréquence. Cette Note donne la construction d'une condition de radiation sur le bord améliorée pour la résolution numérique de l'équation de Helmholtz dans un domaine extérieur pour un régime de hautes fréquences. Cette condition est établie comme approximation de l'opérateur Neumann–Dirichlet en utilisant son symbole principal classique, localement régularisé, dans la zone des rayons glissants afin de les prendre en compte de manière approchée. La simulation de cet opérateur pseudodifférentiel est effectuée efficacement avec un coût linéaire par rapport à la dimension de l'espace d'approximation de type éléments finis frontière grâce à des approximants de Padé complexes adéquats. La méthode est validée numériquement dans le cas d'un obstacle acoustique dur. ***Pour citer cet article : X. Antoine et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

E-mail addresses: antoine@mip.ups-tlse.fr, antoine@acm.caltech.edu (X. Antoine), darbas@mip.ups-tlse.fr (M. Darbas), mayylu@math.cityu.edu.hk (Y.Y. Lu).

1631-073X/$ – see front matter © 2005 Académie des sciences. Published by Elsevier SAS. All rights reserved.
doi:10.1016/j.crma.2005.04.025

Version française abrégée

On propose dans cette Note des améliorations de la technique des conditions de radiation sur le bord, encore appelée en anglais : 'On-Surface Radiation Condition (OSRC) method'. Cette approche, introduite au milieu des années 80 dans un article de Kriegsmann et al. [3], a été développée en vue d'une résolution rapide des problèmes (extérieurs) de diffraction d'ondes en régime fréquentiel (cf. le système (2)). Le principe de base consiste non pas à reformuler de manière équivalente le problème en domaine non borné comme une équation intégrale (exacte), mais plutôt comme une équation aux dérivées partielles de surface (approchée). Tout l'intérêt de cette approche réside dans le fait que le problème de diffraction peut alors être résolu numériquement rapidement. Plus précisément, une stratégie d'approximation par éléments finis frontière conduit à un coût de calcul linéaire par rapport à la dimension N de l'espace fini d'approximation. Ceci est alors bien inférieur au coût d'une méthode intégrale qui est au mieux en $\mathcal{O}(N\log N)$ si une méthode d'accélération de type multipôle multiniveau rapide est utilisée. Le contre-coût à payer est bien sûr lié à la perte de précision de l'approximation obtenue. Celle-ci est clairement en relation avec la fidélité pour laquelle l'opérateur OSRC sous-jacent reproduit la physique du problème.

D'une manière générale, cet opérateur est une approximation de l'opérateur Dirichlet–Neumann (D–N) extérieur. Les approximations existantes ne permettent de modéliser correctement qu'une partie du champ acoustique diffracté au moyen d'une approximation microlocale de l'opérateur D–N dans la zone hyperbolique de l'opérateur [1,2]. Malheureusement, d'autres phénomènes complexes participent également au calcul du champ exact. C'est notamment le cas des rayons glissants et des rayons évanescents. Afin de corriger cet oubli des conditions usuelles, nous proposons, dans la seconde section de cette Note, une nouvelle OSRC qui représente en quelque sorte une approximation uniforme de l'opérateur DtN. Plus précisément, cet opérateur est construit grâce à la connaissance du symbole principal local classique de l'opérateur D–N dans le plan tangent au point considéré. L'opérateur pseudodifférentiel obtenu à partir de ce symbole approche, de manière satisfaisante, l'opérateur D–N dans ses zones hyperbolique et elliptique. En revanche, il ne permet pas de reproduire les phénomènes liés aux rayons glissants. Pour remédier à ce défaut, nous introduisons un paramètre local de dissipation qui régularise l'opérateur approximant. Une estimation optimale de ce paramètre est obtenue. Malgré cette amélioration, l'opérateur OSRC reste encore difficile à implémenter dans un code de calcul puisqu'il est de type non-local car pseudodifférentiel. Nous proposons alors une solution utilisant une approximation de Padé complexe, du symbole d'ordre peu élevé. La condition de radiation de surface obtenue est alors pseudolocale. L'OSRC finale s'écrit sous la forme :

$$C_0\partial_{\mathbf{n}}u+\sum_{j=1}^{p}A_j\frac{\Delta_\Gamma}{k_\varepsilon^2}\left(1+B_j\frac{\Delta_\Gamma}{k_\varepsilon^2}\right)^{-1}\partial_{\mathbf{n}}u=\mathrm{i}k\left(1+\frac{\Delta_\Gamma}{k_\varepsilon^2}\right)u,\quad \text{sur } \Gamma, \tag{1}$$

où les différents coefficients sont donnés par (7) et Δ_Γ désigne l'opérateur de Laplace–Beltrami sur Γ. L'implémentation numérique de (1) est finalement réalisée de manière efficace par une technique d'éléments finis frontière couplée à une résolution itérative par un GMRES avec préconditionneur ILUT. La convergence est extrêmement rapide (2 ou 3 itérations) et confère à l'approche proposée un coût global linéaire par rapport à la dimension de l'espace d'approximation. Enfin nous donnons, dans la Section 3, Fig. 1, un exemple numérique confirmant la qualité de l'approximation obtenue. La méthode OSRC fournit donc une solution approchée précise et efficace pour la résolution des problèmes de diffraction d'ondes. En outre, elle permet de développer de nouvelles formulations intégrales, bien conditionnées, dans le régime de la haute fréquence [2].

1. Introduction

The development of efficient numerical techniques for the simulation of acoustic and electromagnetic scattering problems at high-frequency is of growing interest due to the new requirements in technological and scientific applications. From the point of view of scientific computing, these problems yield challenging questions which require the development of specific and original numerical solutions. Essentially, two well-known difficulties limit the application of classical numerical approaches: the first one is linked to the unbounded character of the exterior domain of propagation and the second one is related to the fact that an asymptotic parameter, the wavelength of the incident field, is small compared to the characteristic size of the scatterer. The past decade has seen the achievement of important improvements as e.g. the development of iterative Krylov integral equation solvers coupled to fast multipole methods or high-order solvers, the simulation of finite element based solvers involving domain decomposition methods and non-reflecting boundary conditions. However, even if these methods are efficient, their application is still limited when high-frequencies are considered. Another solution to these 'classical approximation methods' consists in deriving some suitable Ansatz of the solution of the scattering problem with respect to the asymptotic small parameter, the wavelength. This is the aim, for example, of the Geometric Theory of Diffraction.

An alternative technique, which can be considered in some way at the frontier between 'classical methods' and 'asymptotics techniques', is the On-Surface Radiation Condition (OSRC) method. This approximate approach has been introduced in the middle of the eighties by Kriegsmann et al. [3]. Its aim is roughly to replace the exact non-local surface integral representation of the solution by an approximate local (partial) differential relation. The resulting algorithm has therefore a linear cost according to the dimension of the finite approximation space based on boundary element methods. In regard to this latter remark, this makes the approach very attractive to prospect high-frequency scattering problems. However, one must keep in mind that this approach is only approximate and that its accuracy is closely linked to the choice of the underlying OSRC used in the algorithm. Basically, the OSRC is an approximation of the exterior Dirichlet-to-Neumann (DtN) operator. Therefore, the point of view of microlocal analysis gives some rigorous information and direction concerning the way of constructing an accurate approximate operator. In particular, high-frequency approximations in the hyperbolic zone of the DtN map have been introduced in [1]. Actual existing approximations limit the ability of the OSRC to reproduce complex scattering phenomenons like the modelling of gliding rays at the surface of the body and the incorporation of evanescent modes for the accurate representation of the surface fields. The aim of this Note is to address a possible solution to answer partially to these two latter questions. This Note is organized as follows. In Section 2, we introduce the scattering problem and the OSRC technique. The limitations of the second-order OSRC developed in [1] are explained and the derivation of an improved OSRC is given. Section 3 numerically validates this condition on an example.

2. Improvement of the OSRC technique for the acoustic scattering problem

Let Ω be a bounded domain of $\mathbb{R}^2$ representing an impenetrable body of boundary Γ. We denote by Ω^e the exterior isotropic domain where Ω is embedded. We consider an incident time-harmonic (plane) wave u^{inc} illuminating the scatterer Ω. Then, the diffracted acoustic field u is solution to the following exterior boundary value problem:

$$\begin{aligned}
&\Delta u + k^2 u = 0, \quad \text{in } \Omega^e,\\
&\partial_{\mathbf{n}} u = g, \quad \text{on } \Gamma,\\
&\lim_{|x|\to+\infty} |x|^{1/2}\left(\nabla u \cdot \frac{x}{|x|} - \mathrm{i}ku\right) = 0.
\end{aligned} \tag{2}$$

The wavenumber k in the exterior domain is related to the wavelength λ by: $k = 2\pi/\lambda$. Vector $\mathbf{n}$ is the unit normal vector outwardly directed to Ω. The Neumann boundary condition $g = -\partial_{\mathbf{n}} u^{\mathrm{inc}}$ models a sound-hard scattering problem. Finally, the Sommerfeld radiation condition at infinity is added to get the uniqueness.

Let us assume now that we are able to compute an approximation of the Neumann-to-Dirichlet (NtD) map through an operator Λ: $u \approx \Lambda \partial_{\mathbf{n}} u$, on Γ. This last relation is generally called an On-Surface Radiation Condition

(OSRC). Using the Neumann data, this gives an approximation of the trace on Γ of the unknown field u. Moreover, the Helmholtz integral representation,

$$u(x) = \int_\Gamma \left[\partial_{\mathbf{n}(y)} G(x,y)\partial_{\mathbf{n}} u(y) - G(x,y)u(y)\right] \mathrm{d}\Gamma(y), \quad \text{for } x \in \Omega^e, \tag{3}$$

yields an approximate scattered acoustic field. The Green's function is: $G(x,y) = \mathrm{i}H_0^{(1)}(k|x-y|)/4$, where $H_0^{(1)}$, stands for the first-kind Hankel's function of order zero. Clearly, the accuracy of this process is directly related to the approximation quality of the OSRC operator Λ.

The derivation of the new OSRC starts with the same first step used in [1]. For an arbitrarily shaped boundary Γ, the total symbol of the DtN map Λ^{-1} can be locally expanded as:

$$\sigma_{\Lambda^{-1}}(k,s,\xi) \sim \sum_{j=-1}^{+\infty} \lambda_{-j}(k,s,\xi), \tag{4}$$

where λ_{-j} is a symbol of homogeneous degree $-j$ according to (k,ξ), ξ being the Fourier covariable of the arclength s. In the sequel, we define by $\mathrm{Op}(\sigma)$ the pseudodifferential operator based on the Fourier representation and a given symbol σ. The first four symbols are given in [1] and OSRCs of different orders can be obtained from Taylor expansions of the symbols assuming that the spatial frequencies satisfy $|\xi/k| \ll 1$. For example, the second-order Bayliss–Gunzburger–Turkel-like OSRC [1] corresponds to the symbolical expansions:

$$\begin{aligned}
&\lambda_1 = \mathrm{i}\sqrt{k^2-\xi^2} \approx \mathrm{i}k\left(1-\frac{\xi^2}{2k^2}\right) + \mathcal{O}\left(\frac{1}{k^3}\right), \qquad \lambda_0 \approx -\frac{\kappa}{2}\left(1+\frac{\xi^2}{k^2}\right) + \mathcal{O}\left(\frac{1}{k^3}\right),\\
&\lambda_{-1} \approx \frac{\mathrm{i}\kappa^2}{8k} + \frac{\mathrm{i}}{2}\partial_s\kappa\frac{\xi}{k^2} + \mathcal{O}\left(\frac{1}{k^3}\right), \qquad \lambda_{-2} \approx \frac{\partial_s^2\kappa}{8k^2} + \frac{\kappa^3}{8k^2} + \mathcal{O}\left(\frac{1}{k^3}\right).
\end{aligned} \tag{5}$$

Here, the square-root of a complex number follows the standard definition where the branch-cut is the negative real axis. The exact principal symbol $\lambda_1 = \mathrm{i}\sqrt{k^2-\xi^2}$ defines a non-local pseudodifferential operator. The localization process based on the Taylor expansions gives rise to approximations that are only valid in a conic neighborhood of the hyperbolic zone: $\mathcal{H} = \{(k,\xi) \mid k \gg |\xi|\}$. Therefore, we cannot expect that the process developed in [1] yields OSRCs which model the evanescent modes and creeping rays. From the point of view of microlocal analysis, these two latter zones correspond to the elliptic ($k \ll |\xi|$) and gliding zones ($k \approx |\xi|$) of the operator, respectively.

The expansion (4) is valid both in the conic neighborhoods of the hyperbolic and elliptic regions but is unfortunately non-uniform and breakdowns for the glancing rays corresponding to $k \approx |\xi|$. This means that, even if we consider the following OSRC: $\partial_{\mathbf{n}} u = \mathrm{i}k\,\mathrm{Op}(\sqrt{1-\xi^2/k^2})u$, on Γ, based on the principal classical symbol, the numerical computation of the resulting approximate acoustic field remains inaccurate since the approximate OSRC exhibits a local singularity (unlike a priori the exact DtN operator) in the transition region from the hyperbolic to the elliptic zones. From a numerical point of view, even if this approximation is used, inaccurate computations are still obtained because the glacing rays participate to the far-field pattern. The numerical results presented in [2] corroborate this conclusion.

To circumvent this problem, we propose to use a regularized square-root symbol introducing a positive imaginary part in the wavenumber k inside the square-root. More precisely, we set $k_\varepsilon = k + \mathrm{i}\varepsilon$, for $\varepsilon > 0$, and consider the new boundary condition

$$\partial_{\mathbf{n}} u = \mathrm{i}k\,\mathrm{Op}\left(\sqrt{1-\xi^2/k_\varepsilon^2}\right)u, \quad \text{on } \Gamma. \tag{6}$$

This introduces a local regularization for the tangential modes and the new principal symbol $\mathrm{i}k\sqrt{1-\xi^2/k_\varepsilon^2}$ is now analytic in ξ. An analytic estimate of an optimal parameter ε is proposed in [2]. For a general boundary Γ, we formally have: $\varepsilon_{\text{opt}} = 0.4k^{1/3}\kappa^{2/3}$, where κ is the local curvature.

The boundary condition (6) involves a non-local pseudodifferential operator that is expensive to evaluate directly. A practical approach is to further approximate the square-root symbol by a rational function. The boundary condition can then be evaluated by solving a set of local differential equations defined on Γ. An efficient way to realize it, is to use some complex Padé approximants of order p using the change of variable along the ray $z=-1+r\mathrm{e}^{\mathrm{i}(\pi+\theta)}$ to avoid the branch-cut problem [4] given by:

$$\sqrt{1+X}\approx C_0X+\sum_{j=1}^{p}A_jX(1+B_jX)^{-1},$$

with

$$\begin{aligned}
&C_0=\mathrm{e}^{\mathrm{i}\theta/2}R_p(\mathrm{e}^{-\mathrm{i}\theta}-1), \qquad A_j=\frac{\mathrm{e}^{-\mathrm{i}\theta/2}a_j}{(1+b_j(\mathrm{e}^{-\mathrm{i}\theta}-1))^2}, \qquad B_j=\frac{\mathrm{e}^{-\mathrm{i}\theta}b_j}{1+b_j(\mathrm{e}^{-\mathrm{i}\theta}-1)},\\
&R_p(z)=1+\sum_{j=1}^{p}\frac{a_jz}{1+b_jz}, \qquad a_j=\frac{2}{2p+1}\sin^2\left(\frac{j\pi}{2p+1}\right), \qquad b_j=\cos^2\left(\frac{j\pi}{2p+1}\right).
\end{aligned} \tag{7}$$

The angle θ is to be specified. Practically, the final OSRC (after composition by the square-root operator) can be expressed:

$$C_0\partial_{\mathbf{n}}u+\sum_{j=1}^{p}A_j\frac{\Delta_\Gamma}{k_\varepsilon^2}\left(1+B_j\frac{\Delta_\Gamma}{k_\varepsilon^2}\right)^{-1}\partial_{\mathbf{n}}u=\mathrm{i}k\left(1+\frac{\Delta_\Gamma}{k_\varepsilon^2}\right)u, \quad \text{on } \Gamma, \tag{8}$$

where Δ_Γ is the Laplace–Beltrami operator over Γ.

3. Numerical results

We test the new OSRC for a sound-hard elliptical cylinder E with semi-axes $a=1$ and $b=0.25$ along the x_1- and x_2-directions, respectively. This object is centered at the origin. The incident wave is plane: $u^{\mathrm{inc}}(x)=\exp(ik\nu\cdot x)$, where $\nu=(\cos\theta^{\mathrm{inc}},\sin\theta^{\mathrm{inc}})$ and θ^{inc} is the angle of incidence. In this example, we consider a wavenumber $k=30$ and $\theta^{\mathrm{inc}}=35$ degrees. The reference solution is obtained by an integral equation solver. Numerically, it appears that $\theta=\pi/4$ and $p=8$ correspond to a suitable (stable) choice of the rotating branch-cut approximation.

In Fig. 1, we plot the surface fields (trace) and far-field patterns using the Radar Cross Section (RCS) given by: $\mathrm{RCS}(\vartheta)=10\log_{10}(2\pi|\mathcal{A}(\vartheta)|^2)$, where ϑ is the angle of diffusion and $\mathcal{A}(\vartheta)$ is the scattering amplitude. We observe that a satisfactory computation of these two quantities is obtained. Other numerical simulations [2] on different problems (two- and three-dimensional sound-soft and sound-hard obstacles, two-dimensional scatterers covered by a thin elastic layer and different shapes) confirm this behaviour. Moreover, a general conclusion is that the accuracy of the new OSRC is much better [2] than using the usual OSRC developed for example in [1].

From a computational point of view, the solution of the OSRC problem is efficiently realized by a boundary element method coupled to an ILUT preconditioned iterative GMRES solver [2]. Each of the $(p+1)$ Helmholtz-type surface PDE involved in (8) is solved in only 3 iterations. Therefore, the cost of the overall procedure is $\mathcal{O}((p+1)N)$, if N designates the number of degrees of freedom of the boundary element approximation. This cost is much lower than for example a Fast Multilevel Multipole (FMM) method which yields in the best case to a cost $\mathcal{O}(N\log N)$.

As a conclusion, the OSRC technique based on condition (8) provides a fast and approximate computational way for solving scattering problems in the high-frequency spectrum. This method is particularly interesting when one wants to build new classes of well-posed and well-conditioned generalized Combined Field Integral Equations [2] at high-frequencies. Further applications are actually under progress.

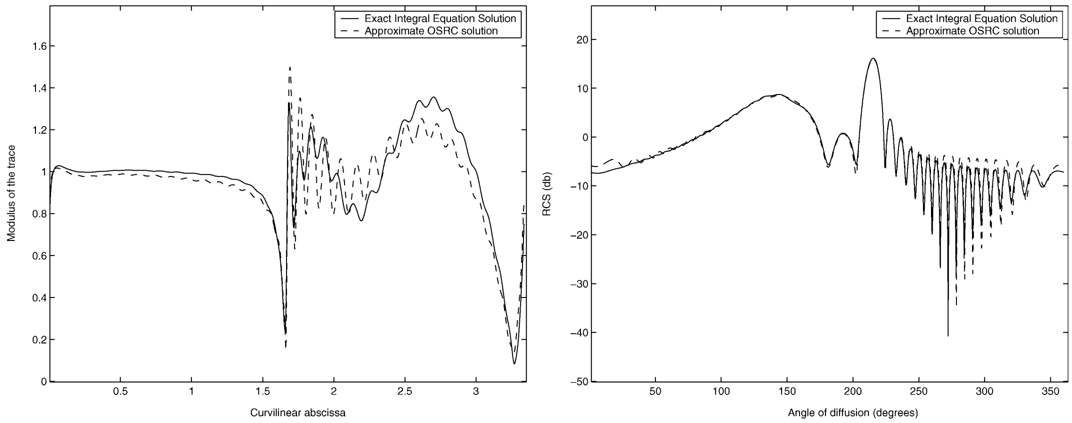

Fig. 1. Sound-hard scatterer E: surface fields and RCS for $k = 30$ and $\theta^{\text{inc}} = 35$ degrees.

Acknowledgements

This work has received the support of the PROCORE-France/Hong-Kong Joint Research Scheme sponsored by the Research Grants Council of Hong-Kong and the Consulate General of France in Hong-Kong (Project No.: F-HK01/03T).

References

[1] X. Antoine, H. Barucq, A. Bendali, Bayliss–Turkel-like radiation conditions on surfaces of arbitrary shape, J. Math. Anal. Appl. 229 (1999) 184–211.
[2] M. Darbas, Préconditionneurs analytiques de type Calderón pour les formulations intégrales des problèmes de diffraction d'ondes, Thèse de Doctorat, INSA, Toulouse, novembre 2004.
[3] G.A. Kriegsmann, A. Taflove, K.R. Umashankar, A new formulation of electromagnetic wave scattering using the on-surface radiation condition method, IEEE Trans. Antennas Propag. 35 (1987) 153–161.
[4] F.A. Milinazzo, C.A. Zala, G.H. Brooke, Rational square-root approximations for parabolic equation algorithms, J. Acoust. Soc. Am. 101 (2) (1997) 760–766.