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Dynamical Systems/Ordinary Differential Equations

A cubic Hénon-like map in the unfolding of degenerate homoclinic orbit with resonance

Marco Martens [a], Vincent Naudot [a], Jiazhong Yang [b,1]

[a] *University of Groningen, Department of Mathematics, P.O. Box 800m, NL-9700 AV Groningen, The Netherlands*
[b] *LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, PR China*

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Abstract

In this Note, we study the unfolding of a vector field that possesses a degenerate homoclinic (of inclination-flip type) to a hyperbolic equilibrium point where its linear part possesses a resonance. For the unperturbed system, the resonant term associated with the resonance vanishes. After suitable rescaling, the Poincaré return map is a cubic Hénon-like map. We deduce the existence of a strange attractor which persists in the Lebesgue measure sense. We also show the presence of an attractor with topological entropy close to $\log 3$. ***To cite this article: M. Martens et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Une application de type Hénon cubique dans le déploiement d'une orbite homocline dégénérée avec résonance. Nous étudions le déploiement d'un champ de vecteurs sur $\mathbb{R}^3$ qui possède une orbite homocline dégénérée associée à une singularité hyperbolique. La partie linéaire du champ en cette singulartité posssède une résonance mais, pour le système initial, le terme résonant associé à cette résonance disparaît. Nous montrons qu'après changement d'échelle, l'application de retour de Poincaré sur une section transverse est proche d' une application de Hénon cubique. Un attracteur étrange est présent et persiste au sens de la mesure de Lebesgue. Nous montrons également la présence d'un attracteur avec une entropie topologique proche de $\log 3$. ***Pour citer cet article : M. Martens et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

E-mail addresses: marco@math.rug.nl (M. Martens), v.naudot@math.rug.nl (V. Naudot), jyang@math.pku.edu.cn (J. Yang).
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1. Introduction

In dynamical systems, from the bifurcation theoretical point of view, homoclinic orbits play an important role. To understand the dynamics that appear after perturbation of such a system, many studies have been done first in the codimension one case [21], but also in the codimension two case [3,4,6,11,18,20] and latter [13,12] in the codimension three case. To really understand the bifurcations around this homoclinic orbit we need more specific information about this orbit. This information can be topological but also analytical and implies in general geometrical considerations [3,4,10,11,14,19]. For instance Homburg et al. study the bifurcations that arise in the unfolding of an *inclination-flip* homoclinic orbit on $\mathbb{R}^3$ and show that a suspended horseshoe is present in a tubular neighbourhood of the unperturbed homoclinic orbit. In this context, the additional degeneracy of the homoclinic orbit comes from a non transversallity condition (see details below). Rychlik [19] shows that a geometrical Lorenz attractor is present in the phase portrait of a three-dimensional vector field that unfolds a double inclination-flip homoclinic orbit with a $\mathbb{Z}_2$ symmetry. Earlier, Robinson [18] shows the same result when the non transversallity condition is replaced by a resonant condition: the sum of two eigenvalues of the linear part of the vector field at the singularity vanishes. The presence of complicated dynamics here is due to the change of the dynamics from attracting to expanding inside the extended unstable manifold. However, the presence of a resonant term associated with the resonance does not contribute to any qualitative change of the dynamic.

In this Note, we propose a scenario similar to that of [11] where the appearance of a resonant term yields to qualitative changes of the dynamics. This will involve four degeneracy conditions.

Let X_p, $p \in \mathcal{D} \subset \mathbb{R}^4$, $0 \in \operatorname{int}(\mathcal{D})$ be a family of smooth vector-field on $\mathbb{R}^3$, with the origin O being a hyperbolic equilibrium point. $DX_p(O)$ has three real eigenvalues $-\alpha(p) < -\beta(p) < 0$ and $\varrho(p) > 0$. We put $\varrho(p) \equiv 1$, this can be obtained by a time rescaling. This implies that O possesses a local stable manifold W^s_{loc} of dimension 2 and a local unstable manifold W^u_{loc} of dimension 1. Since $\alpha(0) > \beta(0)$, there also exists a local strong stable manifold W^{ss}_{loc}, that belongs to the local stable manifold and its tangent space at the equilibrium point is spanned by the eigenspace associated with $-\alpha(p)$. We extend these manifolds by the flow and denote their extension by W^s, W^u and W^{ss} respectively. These manifolds are smooth, unique and invariant under the flow. However, there exists a local invariant manifold $W^{u,s}_{\mathrm{loc}}$ containing W^u_{loc}, called an *extended unstable manifold*, its tangent space at the equilibrium point is spanned by the eigenspaces associated with the eigenvalues $-\beta$ and 1. This manifold is not unique but its tangent space along the unstable manifold does not depend on the choice of the extended unstable manifold. In general $W^{u,s}_{\mathrm{loc}}$ is only C^1 [7]. We shall assume the family X_p to satisfy the following conditions. The first conditions concern the global dynamics of X_0 and the others concern the local dynamics near the origin.

(i) X_0 possesses a non degenerate inclination-flip homoclinic orbit, see below for more details,
(ii) $\alpha(0) = 2\beta(0)$ which is a resonant condition. The associated family of germs then take the following normal form

$$X_p(x, y, z) = Y_p + \lambda(p)z^2\partial/\partial y + G_p(x, y, z), \tag{1}$$

where $Y_p = x\frac{\partial}{\partial x} - \alpha(p)y\frac{\partial}{\partial y} - \beta(p)\frac{\partial}{\partial z}$ is the linear part and $\|G_p(x, y, z)\| = o\|(x, y, z)\|^2$ consists of the higher order terms, see explanations below.
(iii) The resonant coefficient vanishes, i.e., $\lambda(0) = 0$.

Our explanation now concerns condition (i) which says that X_0 possesses a homoclinic orbit $\Gamma = \{\Gamma(t) \mid t \in \mathbb{R}\}$ to the equilibrium point. This orbit is contained in $W^u \cap W^s$. The second degeneracy condition is the non transversallity condition mentioned above and is defined as follows. We say that Γ is an '*inclination-flip*' homoclinic orbit if $W^{u,s}_{\mathrm{loc}}$ is tangent to W^s along Γ. Indeed, this configuration occurs when two smooth functions

$$\varepsilon, \mu : \mathcal{D} \to \mathbb{R}$$

vanish, i.e., X_p possesses an inclination-flip homoclinic orbit if and only if $\varepsilon(p) = 0 = \mu(p)$. Heuristically, these functions are defined as follows. Take a cross section Σ transverse to W^u_{loc}. In Σ, ε stands for the distance between W^u_{loc} and W^s. When this distance vanishes, μ represents the angle between the tangent space of the $W^s \cap \Sigma$ and that of $W^{u,s}_{\text{loc}} \cap \Sigma$. Condition (ii) concerns the eigenvalues. We write $\nu(p) = \alpha(p) - 2\beta(p)$. We assume that $\nu(0) = 0$ i.e., the unperturbed system admits a resonance between the negative eigenvalues. This further implies that the normal form of X_p at the origin takes the form (1). Observe that $z^2 \frac{\partial}{\partial y}$ is a resonant term for Y_0. This explains why we can expand the 2-jet of the family, for $p \sim (0, 0, 0, 0)$ as above. Note that condition (iii) says that the corresponding coefficient vanishes.

Before stating the main result of this paper, we introduce the following notations. Let $\gamma = (\gamma_0, \gamma_1) \in \mathbb{R}^2$, $\delta > 0$. The cubic Hénon map is defined as follows

$$\mathcal{H}^{\pm}_{\gamma,\delta} : \mathbb{R}^2 \to \mathbb{R}^2, \quad (u, v) \mapsto (\gamma_0 + \gamma_1 u \pm u^3 + \delta v, -\delta u),$$

with δ^2 as Jacobian. We say that a family of map $H_{\gamma,\delta} : \mathbb{R}^2 \to \mathbb{R}^2$ is a *cubic Hénon-like family* if the family is close to $\mathcal{H}^{\pm}_{\gamma,0}$ in the C^3 topology of the uniform convergence on compact set and more precisely if there exist $r > 0$ such that

$$\left\| H_{\gamma,\delta} - \mathcal{H}^{\pm}_{\gamma,\delta} \right\|_{C^3} = \mathcal{O}(\delta^r).$$

In what follows, S is a subsection transverse to Γ. The local stable manifold W^s_{loc} split S into two connected components and S^+ is that where the Poincaré return map $\mathcal{P}_p$ associated to X_p is well defined.

Theorem 1.1. *Let $\{X_p, p \in \mathcal{D}\}$ be a family of vector fields that satisfies the above four properties. We moreover assume that $1/3 < \beta(0) < 1/2$ and that the map*

$$\varphi : \mathcal{D} :\to \mathbb{R}^4, \quad p \mapsto \big(\varepsilon(p), \mu(p), \lambda(p), \nu(p)\big)$$

is a diffeomorphism near 0. Let S be a section transverse to Γ and $\mathcal{P}_p : S^+ \to S$ be the Poincaré return map associated to X_p. Then there exists a blow-up

$$\psi : \mathbb{R}^2 \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^4, \quad (\gamma, \delta, \nu) \mapsto \big(\varepsilon(\gamma, \delta, \nu), \mu(\gamma, \delta, \nu), \lambda(\gamma, \delta, \nu), \nu\big)$$

such that, $\varepsilon(\gamma, 0, \nu) \equiv \lambda(\gamma, 0, \nu) \equiv \mu(\gamma, 0, \nu) \equiv 0$. Furthermore, ψ is a diffeomorphism onto its image and for all ν, the family of maps $\mathcal{P}_{\phi^{-1} \circ \psi(\gamma,\delta,\nu)}$ is equal to a cubic Hénon like map after a singular change of coordinates in the section S.

A direct consequence of Theorem 1.1 is that the Poincaré return map is close to a bimodal map. From [5], a strange attractor is present for a set of parameters with positive Lebesgue measure. Recall that a strange attractor

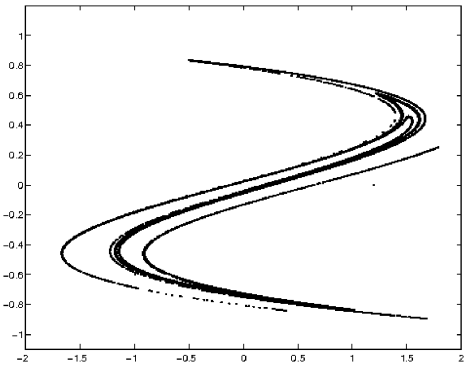

Fig. 1. The attractor for the value $(\gamma_0, \gamma_1) = (0.25, -2.478)$, with $\delta = 0.508$.

Fig. 1. L'attracteur pour les valeurs de $(\gamma_0, \gamma_1) = (0{,}25, -2{,}478)$, $\delta = 0{,}508$.

it possesses a dense orbit, a positive Lyapunov exponent and is non-hyperbolic. Moreover, this attractor is contained in the closure of the unstable manifold of a fixed point of saddle type. Furthermore for δ arbitrarily close to 0, one can construct examples with $\gamma_0 \sim 0$, $\gamma_1 \sim -2$ such that the map $\mathcal{H}^+_{\gamma,\delta}$ possesses an attractor that with entropy close to $\log 3$, which is a topological obstruction for this attractor to be conjugated with the classical Hénon attractor (Fig. 1). This Poincaré return map can be realized in the family X_p. Note that Holmes [8] suggested the cubic Hénon map family as a model for the Poincaré return map associated with the Duffing's equation [9].

2. Sketch of the proof

The proof of Theorem 1.1 is achieved by the following procedure. We fix a subsection $S^+ \subset S$ such that the Poincaré return map $\mathcal{P}_p : S^+ \to S$ is well defined. This latter map is the composition of two maps: the Dulac map which is the transition between S^+ and an intermediate section Σ transverse to the local unstable manifold and a regular map which is the transition map from Σ to S. We use results in [1,2,17] to compute the asymptotics of the Dulac map and then those of the Poincaré return map. This latter, is singular at the intersection with the stable manifold. We then show that for values of the parameter p in $\Phi^{-1} \circ \Psi(\gamma, \delta, \nu)$, for all values of ν the Poincaré return map $\mathcal{P}_{\psi(\gamma,\delta,\nu)}$ is close to the family $\mathcal{H}_{\gamma,\delta}$, after a singular rescaling in the (u, v) coordinates. The techniques developed to construct the blow up generalize those in [15,16].

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