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Partial Differential Equations

Vortices in a 2d rotating Bose–Einstein condensate

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Abstract

We investigate the physical model for a two dimensional rotating Bose–Einstein condensate. We minimize a Gross–Pitaevskii functional defined in $\mathbb{R}^2$ under the unit mass constraint. We estimate the critical rotational speeds Ω_d for having d vortices in the condensate and we determine the location of the vortices. This relies on an asymptotic expansion of the energy. ***To cite this article: R. Ignat, V. Millot, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Tourbillons dans un condensat de Bose–Einstein 2d en rotation. Nous étudions le modèle physique pour un condensat de Bose–Einstein bidimensionnel en rotation. Nous minimisons une fonctionnelle de Gross–Pitaevskii définie sur $\mathbb{R}^2$ sous contrainte de masse un. Nous estimons les vitesses critiques Ω_d pour lesquelles d tourbillons sont présents dans le condensat, puis nous localisons ces tourbillons. Notre méthode est basée sur un développement asymptotique de l'énergie. ***Pour citer cet article : R. Ignat, V. Millot, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Version française abrégée

Le phénomène de condensation de Bose–Einstein a donné lieu à une recherche intense depuis sa première réalisation dans des gaz alcalins en 1995. Un condensat de Bose–Einstein (BEC) est un gaz quantique pouvant être décrit par une seule fonction d'onde complexe. La présence de tourbillons est une particularité majeure de ces systèmes, ils sont définis comme les zéros de la fonction d'onde autour desquels il y a une circulation de phase. Expérimentalement, ces tourbillons peuvent être obtenus par la rotation du piège regroupant les atomes (voir [1,11]). Un modèle bidimensionnel de BEC en rotation a été utilisé par Castin et Dum [7] correspondant à un

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piège harmonique confiné le long de l'axe de rotation. Dans le cas axisymétrique, la fonction d'onde u_ε minimise l'énergie de Gross-Pitaevskii

$$F_\varepsilon(u) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\left[\left(|u|^2 - a(x)\right)^2 - \left(a^-(x)\right)^2\right] - \Omega x^\perp \cdot (\mathrm{i}u, \nabla u) \right\} \mathrm{d}x \tag{1}$$

sous la contrainte de masse

$$\int_{\mathbb{R}^2} |u|^2 = 1 \tag{2}$$

où $\varepsilon > 0$ est un petit paramètre, $\Omega = \Omega(\varepsilon) \geqslant 0$ désigne la vitesse de rotation, $a(x) = a_0 - |x|^2$ et a_0 est determiné par $\int_{\mathbb{R}^2} a^+(x) = 1$, i.e., $a_0 = \sqrt{2/\pi}$.

Notre but est d'étudier le nombre et la position des tourbillons en fonction de la vitesse angulaire $\Omega(\varepsilon)$ quand $\varepsilon \to 0$. Nous nous plaçons dans la situation où Ω est au plus de l'ordre de $|\ln \varepsilon|$. Lorsque $\varepsilon \to 0$, la minimisation de F_ε force $|u_\varepsilon|$ à se rapprocher de $\sqrt{a^+}$. La densité de masse est donc asymptotiquement localisée dans $\mathcal{D} := \{x \in \mathbb{R}^2,\ a(x) > 0\} = B(0, \sqrt{a_0})$. Nous verrons également que $|u_\varepsilon|$ décroît exponentiellement vers 0 en dehors de $\mathcal{D}$. Nous limitons la recherche des tourbillons au disque $\mathcal{D}$. Un développement asymptotique de $F_\varepsilon(u_\varepsilon)$ nous permet d'estimer la vitesse critique Ω_d pour laquelle le d ième tourbillon devient énergétiquement favorable ainsi que l'*énergie renormalisée* gouvernant la position des tourbillons. Nous montrons que ceux-ci sont de taille d'ordre ε et qu'ils sont uniformément repartis près de l'origine à l'échelle $\Omega^{-1/2}$. Les démonstrations complètes des résultats annoncés sont données dans Ignat et Millot [8,9]. Nos résultats peuvent être généralisés pour des potentiels de piégage asymétriques, i.e., $a(x) = a_0 - x_1^2 - \lambda x_2^2$ avec $0 < \lambda < 1$. Les techniques de cette note peuvent aussi être utilisées pour une fonction $a(x)$ plus générale, pour laquelle le domaine où elle est positive n'est plus simplement connexe. Ce problème fera l'objet d'un travail ultérieur.

1. Introduction

The phenomenon of Bose–Einstein condensation has given rise to an intense research since its first realization in alkali gases in 1995. A Bose–Einstein condensate (BEC) is a quantum gas which can be described by a single complex-valued wave function. One of the main feature of these systems is the presence of vortices, i.e., zeroes of the wave function around which there is a circulation of phase. Experimentally vortices can be obtained through the rotation of the trap holding the atoms (see [1,11]). A two-dimensional model for a rotating BEC was used by Castin and Dum [7]. This model corresponds to a harmonic trap strongly confined along the rotation axis. In the case of an axisymmetric trap, the wave function u_ε minimizes the Gross–Pitaevskii energy (1) under the mass constraint (2) where $\varepsilon > 0$ is a small parameter, $\Omega = \Omega(\varepsilon) \geqslant 0$ denotes the rotational velocity, $a(x) = a_0 - |x|^2$ and a_0 is determined by $\int_{\mathbb{R}^2} a^+(x) = 1$ so that $a_0 = \sqrt{2/\pi}$.

Our main goal is to study the number and the location of vortices according to the value of the angular speed $\Omega(\varepsilon)$ as $\varepsilon \to 0$. We consider the situation in which Ω is at most of order $|\ln \varepsilon|$. We will see that in the limit $\varepsilon \to 0$, the minimization of F_ε strongly forces $|u_\varepsilon|^2$ to be closed to a^+ which means that the resulting density is asymptotically localized in $\mathcal{D} := \{x \in \mathbb{R}^2, a(x) > 0\} = B(0, \sqrt{a_0})$. We will also prove that $|u_\varepsilon|$ decays exponentially outside $\mathcal{D}$. We will seek vortices inside the domain $\mathcal{D}$, using an asymptotic expansion of $F_\varepsilon(u_\varepsilon)$.

Let us now recall some related works. In [6], Bethuel, Brezis and Hélein have developed the main tools for studying vortices in "Ginzburg–Landau type" problems. A similar functional to (1) was considered by Serfaty in [13] where $a(x) \equiv 1$ and $\mathbb{R}^2$ is replaced by a disc. She proves the existence of local minimizers having vortices for different ranges of rotational velocity. In [3], Aftalion and Du follow the strategy in [13] for the study of global minimizers of the Gross–Pitaevskii energy (1) where $\mathbb{R}^2$ is replaced by $\mathcal{D}$. In [2], Aftalion, Alama and Bronsard analyze the global minimizers of (1) for potentials of different nature leading to an annular region of confinement.

We emphasize that we tackle here the problem which corresponds exactly to the physical model. In particular, we minimize F_ε under the mass constraint (2) and the admissible configurations are defined on the whole space $\mathbb{R}^2$. Several difficulties arise, especially in the proof of the existence results and the construction of test functions. We point out that we do not assume any implicit bound on the number of vortices. The singular and degenerate behavior of $\sqrt{a^+}$ near $\partial\mathcal{D}$ induces a cost of order $|\ln\varepsilon|$ in the energy and requires specific tools to detect vortices in the boundary region.

2. Main results

We now start to describe our main results. We introduce the functional space in which we perform the minimization $\mathcal{H} := \{u \in H^1(\mathbb{R}^2, \mathbb{C}), \int_{\mathbb{R}^2} |x|^2|u|^2 < \infty\}$. When $\Omega = 0$, $F_\varepsilon(u) = E_\varepsilon(u)$ where

$$E_\varepsilon(u) = \int_{\mathbb{R}^2} \left\{\frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\left[\left(|u|^2 - a(x)\right)^2 - \left(a^-(x)\right)^2\right]\right\} \mathrm{d}x. \tag{3}$$

We shall prove that for ε small enough, the minimization problem

$$\mathrm{Min}\left\{E_\varepsilon(\eta), \eta \in \mathcal{H}, \|\eta\|_{L^2(\mathbb{R}^2)} = 1\right\} \tag{4}$$

admits a unique solution $\tilde{\eta}_\varepsilon$ (up to a complex multiplier of modulus one) which is a real positive function. Moreover, $\tilde{\eta}_\varepsilon$ converges to $\sqrt{a^+}$ in $L^\infty(\mathbb{R}^2)$ as $\varepsilon \to 0$. Defining for any integer $d \geqslant 1$, the critical velocities

$$\Omega_d = \frac{2}{a_0}|\ln\varepsilon| + \frac{2(d-1)}{a_0}\ln|\ln\varepsilon|, \tag{5}$$

our main theorem can be stated as follows:

Theorem 2.1. *Let u_ε be any minimizer of F_ε in $\mathcal{H}$ under the mass constraint (2) and let $0 < \delta \ll 1$ be any small constant.*

(i) *If $\Omega \leqslant \Omega_1 - \delta\ln|\ln\varepsilon|$, then for any $R_0 < \sqrt{a_0}$, there exists $\varepsilon_{R_0} > 0$ such that for any $\varepsilon < \varepsilon_{R_0}$, u_ε is vortex free in B_{R_0}, i.e., u_ε does not vanish in B_{R_0}. In addition,*

$$F_\varepsilon(u_\varepsilon) = E_\varepsilon(\tilde{\eta}_\varepsilon) + \mathrm{o}(1).$$

(ii) *If $\Omega_d + \delta\ln|\ln\varepsilon| \leqslant \Omega \leqslant \Omega_{d+1} - \delta\ln|\ln\varepsilon|$ for some integer $d \geqslant 1$, then for any $R_0 < \sqrt{a_0}$, there exists $\varepsilon_{R_0} > 0$ such that for any $\varepsilon < \varepsilon_{R_0}$, u_ε has exactly d vortices $x_1^\varepsilon, \ldots, x_d^\varepsilon$ of degree one in B_{R_0}. Setting $\tilde{x}_j^\varepsilon = \sqrt{\Omega}x_j^\varepsilon$, the configuration $(\tilde{x}_1^\varepsilon, \ldots, \tilde{x}_d^\varepsilon)$ tends to minimize in $\mathbb{R}^{2d}$ the renormalized energy*

$$w(b_1, \ldots, b_d) = -\pi a_0 \sum_{i \neq j} \ln|b_i - b_j| + \frac{\pi a_0}{2}\sum_{j=1}^{d}|b_j|^2$$

so that $|x_j^\varepsilon| \leqslant C\Omega^{-1/2}$ for any $j = 1, \ldots, d$ and $|x_i^\varepsilon - x_j^\varepsilon| \geqslant C\Omega^{-1/2}$ for any $i \neq j$ for some constant $C > 0$ independent of ε. In addition,

$$F_\varepsilon(u_\varepsilon) = E_\varepsilon(\tilde{\eta}_\varepsilon) - \frac{\pi a_0^2}{2}d(\Omega - \Omega_1) + \frac{\pi a_0}{2}(d^2 - d)\ln|\ln\varepsilon| + \mathrm{Min}_{b\in\mathbb{R}^{2d}}\, w(b) + Q_d + \mathrm{o}(1) \tag{6}$$

where Q_d is an explicit constant depending only on d.

These results are in agreement with experimental observations and theoretical predictions on Bose–Einstein condensates. More precisely, the critical angular velocity Ω_1 coincides with that found in [3,7] and the vortices are concentrated around the origin at a scale $\sqrt{\Omega}$.

Our results can be extended to the case of asymmetric trapping potentials, i.e., $a(x) = a_0 - x_1^2 - \lambda x_2^2$ with $0 < \lambda < 1$. The techniques of this Note can also be used in the case where the function $a(x)$ is positive in a domain which is not simply connected. This will be the topic of a future work.

3. Sketch of the proof

We now outline the main steps of the proof of Theorem 2.1 (for the detailed proof, we refer to Ignat and Millot [8,9]). We first prove the existence of minimizers u_ε under the mass constraint (2) and some general results about their behavior: $E_\varepsilon(u_\varepsilon) \leqslant C|\ln\varepsilon|^2$, $|\nabla u_\varepsilon| \leqslant C_K\varepsilon^{-1}$ and $|u_\varepsilon| \lesssim \sqrt{a^+}$ in any compact $K \subset \mathcal{D}$, and u_ε decreases exponentially quickly to 0 outside $\mathcal{D}$. For the study of the density profile $\tilde{\eta}_\varepsilon$ defined by (4), we introduce the real positive minimizer η_ε of E_ε, i.e., $E_\varepsilon(\eta_\varepsilon) = \min_{\eta\in\mathcal{H}} E_\varepsilon(\eta)$. We show the existence and uniqueness of η_ε and we have that $\eta_\varepsilon \to \sqrt{a^+}$ in $L^\infty(\mathbb{R}^2) \cap C^1_{\mathrm{loc}}(\mathcal{D})$ as $\varepsilon \to 0$. Then we explicitly characterize the link between η_ε and $\tilde{\eta}_\varepsilon$ and we prove that $|E_\varepsilon(\eta_\varepsilon) - E_\varepsilon(\tilde{\eta}_\varepsilon)| = \mathrm{o}(\varepsilon)$. Since the mass of η_ε may not be equal to 1 in general, we will use the profile $\tilde{\eta}_\varepsilon$ as a test function.

The second step concerns the decoupling of the energy. As in [2,10,13], we prove that $F_\varepsilon(u_\varepsilon)$ splits into two independent pieces, the energy $E_\varepsilon(\eta_\varepsilon)$ and a reduced energy $\mathcal{F}_\varepsilon^{\eta_\varepsilon}$ of $v_\varepsilon = u_\varepsilon/\eta_\varepsilon$, i.e.,

$$F_\varepsilon(u_\varepsilon) = E_\varepsilon(\eta_\varepsilon) + \mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) \tag{7}$$

with

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) = \mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) - \mathcal{R}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon), \qquad \mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) = \int_{\mathbb{R}^2} \frac{\eta_\varepsilon^2}{2}|\nabla v_\varepsilon|^2 + \frac{\eta_\varepsilon^4}{4\varepsilon^2}\left(|v_\varepsilon|^2 - 1\right)^2,$$

$$\mathcal{R}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) = \Omega \int_{\mathbb{R}^2} \eta_\varepsilon^2 x^\perp \cdot (\mathrm{i}v_\varepsilon, \nabla v_\varepsilon).$$

In (7), $E_\varepsilon(\eta_\varepsilon)$ carries the energy of the singular layer near $\partial\mathcal{D}$ and hence, we may detect vortices by the reduced energy $\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon)$. We study the vortex structure of u_ε via v_ε applying the Ginzburg–Landau techniques to the weighted energy $\mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon)$; the difficulty will arise in the region where η_ε is small. We notice that v_ε inherits the following properties: $\mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) \leqslant C|\ln\varepsilon|^2$, $|\nabla v_\varepsilon| \leqslant C_K\varepsilon^{-1}$ and $|v_\varepsilon| \lesssim 1$ in any compact $K \subset \mathcal{D}$. Using $\tilde{\eta}_\varepsilon$ as a test function, we obtain by (7),

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, \mathcal{D}) \leqslant \mathrm{o}(1). \tag{8}$$

Now we compute a first lower estimate of $\mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon)$ using a method due to Sandier and Serfaty (see [12]). We start with a first construction of small vortex balls $\{B(p_i, r_i)\}_{i\in I_\varepsilon}$ in a domain $\mathcal{D}_\varepsilon$ slightly smaller than $\mathcal{D}$: outside these balls $|v_\varepsilon|$ is close to 1, so that v_ε carries a degree d_i on $\partial B(p_i, r_i)$ and

$$\mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, \mathcal{D}_\varepsilon) \geqslant \sum_{i\in I_\varepsilon} \mathcal{E}_\varepsilon^{\eta_\varepsilon}\big(v_\varepsilon, B(p_i, r_i)\big) \gtrsim \pi \sum_{i\in I_\varepsilon} a(p_i)|d_i||\ln\varepsilon|. \tag{9}$$

Then we prove an asymptotic expansion of the rotational energy outside the balls $\{B(p_i, r_i)\}_{i\in I_\varepsilon}$,

$$\mathcal{R}_\varepsilon^{\eta_\varepsilon}\Bigg(v_\varepsilon, \mathcal{D}_\varepsilon \Big\backslash \bigcup_{i\in I_\varepsilon} B(p_i, r_i)\Bigg) \approx \frac{\pi\Omega}{2} \sum_{i\in I_\varepsilon} a^2(p_i) d_i. \tag{10}$$

Estimates (9) and (10) yield a first lower bound of $\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, \mathcal{D})$ that we match with (8) in order to derive the first critical angular velocity Ω_1 and to prove the absence of vortices for velocities strictly less than Ω_1. We also obtain that for $\Omega \leqslant \Omega_1 + \mathcal{O}(\ln|\ln\varepsilon|)$, the number of vortex balls with nonzero degree is uniformly bounded in ε and they are concentrated around the origin. We conclude by

$$\mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, \mathcal{D}_\varepsilon) = \mathcal{O}\big(|\ln\varepsilon|\big) \quad \text{and} \quad \mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, \mathcal{D}_\varepsilon \setminus B_{2|\ln\varepsilon|^{-1/6}}) = \mathrm{o}\big(|\ln\varepsilon|\big). \tag{11}$$

In the following, we give a finer description of the vortex structure in $B_R \subset\subset \mathcal{D}$ using the method of "bad discs" introduced by Bethuel, Brezis and Hélein [6]. We find that the number of bad discs is uniformly bounded, all of them remaining close to the origin. The main ingredients are the energy estimates in (11) and a local version of the Pohozaev identity. Using the "clustering" method presented in [4], we obtain a new family of modified bad discs $\{B(x_j^\varepsilon, \rho)\}_{j\in J_\varepsilon}$ such that $\rho \sim \varepsilon^\alpha$, $0<\alpha<1$, $|v_\varepsilon| \geqslant 1/2$ outside these discs and v_ε has a nonzero degree D_j on each $\partial B(x_j^\varepsilon, \rho)$. We identify *vortices* with the points x_j^ε's.

The next step is to find lower estimates for the energy taking into account the interaction between vortices. Following similar methods to [6], we evaluate the energy carried by each vortex

$$\mathcal{E}_\varepsilon^{\eta_\varepsilon}\big(v_\varepsilon, B(x_j^\varepsilon, \rho)\big) \geqslant \pi a(x_j^\varepsilon)|D_j| \ln\frac{\rho}{\varepsilon} + \mathcal{O}(1) \tag{12}$$

and the energy away from the vortices

$$\mathcal{E}_\varepsilon^{\eta_\varepsilon}\Bigg(v_\varepsilon, B_R \Big\backslash \bigcup_{j\in J_\varepsilon} B(x_j^\varepsilon, \rho)\Bigg) \geqslant \pi \sum_{j\in J_\varepsilon} D_j^2 a(x_j^\varepsilon)|\ln\rho| + W_{R,\varepsilon}\big((x_j^\varepsilon, D_j)_{j\in J_\varepsilon}\big) + \mathcal{O}_R(1). \tag{13}$$

Here, the radius $R \in (\frac{\sqrt{a_0}}{2}, \sqrt{a_0})$ is fixed and the error term $\mathcal{O}_R(1)$ is computed in function of R. The quantity $W_{R,\varepsilon}$ is similar to the renormalized energy in [6] and involves the interaction between the vortices. As for (10), we compute an asymptotic expansion of $\mathcal{R}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \bigcup_{j\in J_\varepsilon} B(x_j^\varepsilon, \rho))$ and it yields

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, \mathcal{D}_\varepsilon) \geqslant \mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, B_R) - \frac{\pi\Omega}{2}\sum_{j\in J_\varepsilon} a^2(x_j^\varepsilon) D_j + \mathrm{o}_R(1). \tag{14}$$

Combining (8), (12), (13) and (14), we deduce that every vortex is of degree 1, i.e., $D_j = 1$, which allows us to improve the above estimates and to obtain the result in the subcritical case (i) in Theorem 2.1.

The final step is to compute an upper bound of $\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon)$. We construct appropriate test functions adapting a method due to André and Shafrir [5]. We are then led to the following expansion

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) = -\frac{\pi a_0^2}{2} n(\Omega - \Omega_1) + \frac{\pi a_0}{2}(n^2 - n)\ln|\ln\varepsilon| + \mathcal{O}(1) \quad \text{where } n = \operatorname{Card} J_\varepsilon.$$

If $\Omega_d + \delta\ln|\ln\varepsilon| \leqslant \Omega \leqslant \Omega_{d+1} - \delta\ln|\ln\varepsilon|$ for any small $\delta > 0$, this expansion yields the exact number of vortices: $n = d$. Moreover, we find that the vortices are uniformly distributed at a scale $\Omega^{-1/2}$ around the origin. Then we compute an asymptotic formula of $W_{R,\varepsilon}$ as $\varepsilon \to 0$ and $R \to \sqrt{a_0}$:

$$W_{R,\varepsilon}(x_1^\varepsilon, \ldots, x_d^\varepsilon) = -\pi a_0 \sum_{i\neq j} \ln|x_i^\varepsilon - x_j^\varepsilon| - \frac{\pi a_0 d^2}{2}(1 - \ln a_0) + \mathrm{o}(1).$$

Using the upper bound of $\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon)$ given by the test functions, we conclude that the rescaled configuration $(\tilde{x}_1^\varepsilon, \ldots, \tilde{x}_d^\varepsilon)$ tends to minimize the renormalized energy w and we also find the expansion of the energy (6).

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