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Differential Geometry

Closed hypersurfaces of $\mathbb{S}^4(1)$ with constant mean curvature and zero Gauß–Kronecker curvature

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Abstract

We consider a closed hypersurface $M^3 \subset \mathbb{S}^4(1)$ with identically zero Gauß–Kronecker curvature. We prove that if M^3 has constant mean curvature H, then M^3 is minimal, i.e., $H = 0$. This result extends Ramanathan's classification (Math. Z. 205 (1990) 645–658) result of closed minimal hypersurfaces of $\mathbb{S}^4(1)$ with vanishing Gauß–Kronecker curvature. ***To cite this article: T. Lusala, A. Gomes de Oliveira, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Hypersurfaces fermées de $\mathbb{S}^4(1)$ à courbure moyenne constante et à courbure de Gauß–Kronecker nulle. Nous considérons une hypersurface fermée (compacte et sans bord) $M^3 \subset \mathbb{S}^4(1)$ à courbure de Gauß–Kronecker identiquement nulle. Nous prouvons que si la courbure moyenne H de M^3 est constante, alors l'hypersurface M^3 est necéssairement minimale, c.à.d, $H = 0$. Ce résultat généralise celui obtenu dans l'article de Ramanathan (Math. Z. 205 (1990) 645–658) concernant les hypersurfaces fermées minimales à courbure de Gauß–Kronecker identiquement nulle dans $\mathbb{S}^4(1)$. ***Pour citer cet article : T. Lusala, A. Gomes de Oliveira, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

1. Introduction

Let $M^3 \subset \mathbb{S}^4(1)$ be a closed hypersurface in the unit Euclidean sphere $\mathbb{S}^4(1)$. Denote by H, σ_2 and K, the mean curvature, the second elementary symmetric function and the Gauß–Kronecker curvature function of M^3, respectively. Almeida and Brito [2] proposed to classify the closed hypersurface M^3 when two of its three curvature

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functions H, σ_2 and K are constant. The survey of results in [4] shows that the case when K is constant is not yet completely solved. In particular for $K \equiv const \neq 0$ and $H \equiv 0$, Almeida and Brito [1] proved that M^3 must be an isoparametric hypersurface; and for $H \equiv const \neq 0$, they obtained the same conclusion [2] but under an additional condition that $\frac{H}{K} \geqslant -3$. This technical condition has been recently removed in [3]. Therefore the case $K \equiv 0$ and $H \equiv const \neq 0$ remains of interest. Ramanthan [10] gave a complete classification of closed minimal hypersurface of $\mathbb{S}^4(1)$ with zero Gauß–Kronecker curvature. Namely he proved the following classification result:

Theorem 1.1 (Ramanathan [10]). *Let M^3 be a compact orientable 3-dimensional manifold and $x : M \to \mathbb{S}^4(1)$ a minimal hypersurface immersion of M^3. If the Gauß–Kronecker curvature of M^3 is identically zero, then either*

(i) $M^3 = \mathbb{S}^3(1)$ *or*
(ii) *there exist a minimal immersion $g : N^2 \to \mathbb{S}^4(1)$ of a compact surface N^2 and a map $\tau : M^3 \to N_g$ such that $x = x_g \circ \tau$, where $N_g = \{(p, \vec{v}) \in N^2 \times \mathbb{R}^5\colon \|\vec{v}\| = 1,\ \vec{v} \perp \mathbb{R} \cdot g(p) + g_*(T_p N^2)\}$ is the unit normal bundle of the immersion g and $x_g : N_g \to \mathbb{S}^4(1)$ is the projection to the second factor.*

In this short paper, we show that instead of the minimality assumption in this classification result, one can consider that the mean curvature of the closed hypersurface with identically zero Gauß–Kronecker curvature is constant. Namely we prove (main result)

Theorem 1.2. *Let $M^3 \subset \mathbb{S}^4(1)$ be a closed hypersurface immersed in $\mathbb{S}^4(1)$ with identically zero Gauß–Kronecker curvature. If M^3 has constant mean curvature, then M^3 is a minimal hypersurface.*

This provides, using Ramanathan's result, a complete classification of closed hypersurfaces of $\mathbb{S}^4(1)$ with constant mean curvature and identically zero Gauß–Kronecker curvature.

Remark 1. If the rank of the second fundamental form of a closed hypersurface M^3 minimally immersed into $S^4(1)$ with identically zero Gauß–Kronecker curvature is constant (equal to 2), Theorem 1.1 was proved in [1]. In this case, M^3 is a boundary of a tube which is built over a non-degenerate minimal 2-dimensional surface immersion in $\mathbb{S}^4(1)$ with geodesic radius $\frac{\pi}{2}$.

2. Notations and facts

Let $x : M^3 \to \mathbb{S}^4(1)$ be a 3-dimensional immersed hypersurface in the unit Euclidean 4-sphere $\mathbb{S}^4(1)$. Let $\{e_1, \ldots, e_4\}$ be a local orthonormal frame fields of $\mathbb{S}^4(1)$ such that e_1, e_2 and e_3 are tagential to M^3, $\{\omega_1, \ldots, \omega_4\}$ the corresponding dual frame and $\{\omega_{ij}\}$ be the connection 1-forms. The structure equations of $\mathbb{S}^4(1)$ are given by

$$\mathrm{d}\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \quad \mathrm{d}\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \overline{R}_{ABCD} \omega_C \wedge \omega_D,$$

where $\overline{R}_{ABCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}$ defines the curvature tensor of $\mathbb{S}^4(1)$. Now we restrict all tensors to M^3. Because $\omega_4 = 0$, we have $\sum_i \omega_{4i} \wedge \omega_i = d\omega_4 = 0$. By Cartan's lemma, we have $\omega_{4i} = \sum_j h_{ij}\omega_j$, with $h_{ij} = h_{ji}$. The tensor $h = \sum_{i,j} h_{ij}\omega_i\omega_j$ is the so called second fundamental form. The eigenvalues λ_i of the matrix (h_{ij}) are the principal curvatures. The elementary functions $H = \frac{1}{3}\mathrm{trace}(h_{ij}) = \sum_i \lambda_i$, $S = \sum_{i,j} h_{ij}^2 = \sum_i \lambda_i^2$ and $K = \det(h_{ij}) = \prod_i \lambda_i$, are known to be the mean curvature, the square of the length of the second fundamental form and the Gauß–Kronecker curvature of M^3, respectively. The restricted structure equations on M^3 imply the following integrability conditions (Gauß and Codazzi equations): $R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk})$, $h_{ijk} = h_{ikj}$, where R is the curvature tensor of M^3, and the covariant derivative h_{ijk} of h_{ij} is defined by $\sum_k h_{ijk}\omega_k = dh_{ij} +$

$\sum_k h_{kj}\omega_{ki} + \sum_k h_{ki}\omega_{kj}$. Let f_k $(k = 3, 4)$ denote the smooth function on M^3 defined by $f_k = \sum_{i=1}^{3} \lambda_i^k$. Defining the functions $\mu_i := \lambda_i - H$, we have that $\sum_i \mu_i = 0$. The following classical formulas for the Laplacians of S and f_3 are well known and can be found in many papers such as [5–9]:

$$\frac{1}{2}\Delta S = (3 - S)S - 9H^2 + 3Hf_3 + \sum_{i,j,k} h_{ijk}^2, \tag{1}$$

$$\frac{1}{3}\Delta f_3 = (3 - S) f_3 + 3Hf_4 - 3HS + 2\sum_{i,j,k} \lambda_i h_{ijk}^2. \tag{2}$$

3. Proof of the main result

Assume from now on that the closed hypersurface M^3 has constant mean curvature and vanishing Gauß–Kronecker curvature function. In this case the characteristic polynomial of the matrix (h_{ij}) is given by $p(\lambda) = \lambda^3 - 3H\lambda^2 + \frac{1}{2}(9H^2 - S)\lambda$. Because the principal curvatures are real, we have that $2S - 9H^2 \geqslant 0$ everywhere, in particular we have that $\min S \geqslant \frac{9}{2}H^2$.

It also follows that the functions f_3 and f_4 can be expressed in terms of H and S:

$$f_3 = \frac{9}{2}H(S - 3H^2), \quad \text{and} \quad f_4 = -\frac{81}{2}H^4 + \frac{1}{2}S^2 + 9H^2 S. \tag{3}$$

Since the mean curvature H is constant, we can write

$$\frac{1}{3}\Delta f_3 = 3H\left(\frac{1}{2}\Delta S\right). \tag{4}$$

Using the expressions (1) and (2) of the Laplacians of S and f_3, the following equation can be deduced from Eq. (4):

$$27H^3 + (3 - S - 9H^2) f_3 + 3Hf_4 + 3HS(S - 4) + 2\sum_{i,j,k} \mu_i h_{ijk}^2 + 3H\sum_{i,j,k} h_{ijk}^2 = 0.$$

Now take a maximum point p of S in M^3. Since $K \equiv 0$, we can assume that $\lambda_3(p) \equiv \lambda_3 = 0$. Suppose that $\lambda_1 = \lambda_2$ at p. In this case, we have that $\lambda_1 = \lambda_2 = \frac{3}{2}H$. So $\max S = S(p) = \frac{9}{2}H^2 = \min S$. This implies that S is constant. Therefore M^3 is isoparametric with at most two distinct principal curvatures, thus M^3 is the totally geodesic great sphere $\mathbb{S}^3(1)$. Suppose that $\lambda_1 \neq \lambda_2$ at p. If $\lambda_2(p) = 0$ (similar case if we consider $\lambda_1(p) = 0$), then $\lambda_1(p) = 3H$. So $S(p) = 9H^2$ and $f_3(p) = 27H^3$. We have that

$$0 \geqslant \frac{1}{2}\Delta S(p) = (3 - 9H^2) - 9H + 81H^4 + \sum_{i,j,k} h_{ijk}^2 = 18H^2 + \sum_{i,j,k} h_{ijk}^2 \geqslant 0,$$

implying in particular that $H = 0$. This is a contradiction since $H = \frac{1}{3}\lambda_1(p) \neq \lambda_2(p) = 0$. Therefore the three principal curvatures have to be distinct at p if $\lambda_1 \neq \lambda_2$. In this last case, we want to prove that M^3 must only be minimal. We have at p for any k:

$$h_{11k} + h_{22k} + h_{33k} = 0 \quad (H \equiv const),$$

$$\lambda_1 h_{11k} + \lambda_2 h_{22k} = 0 \quad (\nabla S(p) = 0 \quad \text{and} \quad \lambda_3(p) = 0),$$

$$\lambda_1^2 h_{11k} + \lambda_2^2 h_{22k} = 0 \quad \left(\nabla f_3(p) = \frac{9}{2}H\nabla S(p) = 0 \quad \text{and} \quad \lambda_3(p) = 0\right).$$

Because the three principal curvatures are distinct at p, we have $h_{iik}=0$ at p for any i, k. It follows that at p, $\sum_{i,j,k} h_{ijk}^2 = 6h_{123}^2$ and $\sum_{i,j,k} \mu_i h_{ijk}^2 = 2(\mu_1+\mu_2+\mu_3)h_{123}^2 = 0$. Hence,

$$27H^3 + (3-S-9H^2)f_3 + 3Hf_4 + 3HS(S-4) + 18Hh_{123}^2 = 0.$$

The insertion of the expressions (3) of f_3 and f_4 into the equation above provides

$$3H\left(6h_{123}^2(p) + \frac{1}{2}\big(S(p)-9H^2\big)\right) = 0.$$

Therefore $H=0$, i.e, M^3 is minimal. Otherwise, we have $6h_{123}^2(p) = \frac{1}{2}(9H^2 - S(p))$. To finish the proof, we have to show that this later case cannot occur. Suppose that $H \neq 0$. In this case we get an upper bound for S: $\frac{9}{2}H^2 \leqslant S \leqslant 9H^2$. The Laplacian of S at the maximum point p is given by

$$0 \geqslant \frac{1}{2}\Delta S(p) = (3-S)S(p) - 9H^2 + 3Hf_3 + \sum_{i,j,k} h_{ijk}^2 = \frac{1}{2}(5+27H^2)S(p) - \frac{9}{2}H^2 - \frac{81}{2}H^4 - S^2(p).$$

This provides the following second order polynomial inequality in $S(p)$ with constant coefficients (depending only on the constant H): $S^2(p) - \frac{1}{2}(5+27H^2)S(p) + \frac{9}{2}H^2 + \frac{81}{2}H^4 \geqslant 0$. Therefore, $S(p) \leqslant S_-(p)$ or $S(p) \geqslant S_+(p)$, where $S_\pm(p) = \frac{5}{4} + \frac{27}{4}H^2 \pm \frac{1}{4}\sqrt{25+198H^2+81H^4}$. If $S(p) \leqslant S_-(p)$, then we have $\frac{9}{2}H^2 \leqslant S(p) \leqslant S_-(p) < \frac{9}{2}H^2$. This is absurd. Also if $S(p) \geqslant S_+(p)$, then we have $9H^2 \geqslant S(p) \geqslant S_+(p) > \frac{5}{4} + 9H^2$, which is impossible. This completes the proof.

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