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Calculus of Variations

# A theory of anti-selfdual Lagrangians: dynamical case

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**Abstract**

We consider the class of *time-dependent anti-selfdual Lagrangians*, which – just like the stationary case announced in Ghoussoub [C. R. Acad. Sci. Paris, Ser. I 340 (2005)] – enjoys remarkable permanence properties and provides variational formulations and resolutions for several initial-value parabolic equations including gradient flows and other dissipative systems. Even though these evolutions do not fit in the standard Euler–Lagrange theory, we show that their solutions can be obtained as minima – but also as zeroes – of action functionals of the form $\int_0^T L(t, u(t), \dot{u}(t) + \Lambda_t u(t)\,\mathrm{d}t)$ where $L$ is a time-dependent anti-selfdual Lagrangian and where $\Lambda_t$ is a flow of skew-adjoint operators. ***To cite this article: N. Ghoussoub, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***


**Résumé**

**Lagrangiens anti-autoduaux : le cas dynamique.** On considère le cas des lagrangiens anti-autoduaux qui dépendent du paramètre temps. Comme dans le cas stationnaire annoncé dans Ghoussoub [C. R. Acad. Sci. Paris, Ser. I 340 (2005)], cette classe possède des propriétés de permanence remarquables qui permettent une formulation et une résolution variationnelle de plusieurs équations paraboliques dissipatives qui ne sont pas normalement de type Euler–Lagrange. ***Pour citer cet article : N. Ghoussoub, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***


## Version française abrégée

À tout lagrangien anti-autodual autonome $L$ sur $X \times X^*$ (où $X$ est reflexif), on associe un semi-group de contractions $(T_t)_{t\in\mathbf{R}^+}$ tel que $x(t) = T_t x$ est la solution de $(-\dot{x}(t), -x(t)) \in \partial L(x(t), \dot{x}(t))$ avec $x(0) = x$. On

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associe un nouveau principe variationnel à une classe importante d'équations – ainsi que des inéquations – paraboliques dissipatives. Les solutions sont obtenues comme minima – mais aussi surtout comme des racines – de fonctionnelles d'action de la forme $\int_0^T L(t,u(t),\dot u(t)+\Lambda_t u(t))\,\mathrm{d}t$, où $L$ est un Lagrangien anti-autodual et où $\Lambda_t$ est un flow d'opérateurs antisymmétriques. Ces équations peuvent être des flots de gradients à potentiel convexe, comme dans l'équation de la chaleur et celle des médias poreux, mais aussi des évolutions nonlinéaires associées à des opérateurs du premier ordre, et donc non-autoadjoints.

## 1. Time-dependent anti-selfdual Lagrangians

Let $H$ be a Hilbert space and let $[0,T]$ be a fixed real interval. Consider the space $L^2_H$ of integrable functions from $[0,T]$ into $H$ with norm denoted by $\|\cdot\|_2$, as well as the Hilbert space $A^2_H=\{u\colon [0,T]\to H;\ \dot u\in L^2_H\}$ consisting of all absolutely continuous arcs equipped with the norm $\|u\|_{A^2_H}=(\|u(0)\|_H^2+\int_0^T\|\dot u\|^2\,\mathrm{d}t)^{1/2}$.

Motivated by the resolution of the Brezis–Ekeland conjectures in [1,3,4,6] we introduce the following.

**Definition 1.1.** Let $L:[0,T]\times H\times H\to\mathbf{R}\cup\{+\infty\}$ be measurable with respect to the $\sigma$-field generated by the products of Lebesgue sets in $[0,T]$ and Borel sets in $H\times H$. Say that $L$ is an anti-selfdual Lagrangian (*ASD*) on $[0,T]\times H\times H$ if $L_t:(x,p)\to L(t,x,p)$ is in $\mathcal{L}_{\mathrm{AD}}(H)$ for any $t\in[0,T]$: that is if $L^*(t,p,x)=L(t,-x,-p)$ for all $(x,p)\in H\times H$, where $L^*$ is the Legendre transform in the last two variables.

**Definition 1.2.** Say that a convex lower semi-continuous function $\ell:H\times H\to\mathbf{R}\cup\{+\infty\}$ is a selfdual time-boundary Lagrangian if $\ell^*(-h_1,h_2)=\ell(h_1,h_2)$ for all $(h_1,h_2)\in H\times H$.

A remarkable permanence property of ASD Lagrangians is that they 'lift' to path spaces.

**Proposition 1.1.** *Suppose that $L$ is an anti-selfdual Lagrangian on $[0,T]\times H\times H$, then*

1. *For each $\omega\in\mathbf{R}$, the Lagrangian $M(u,p):=\int_0^T e^{2wt}L(t,e^{-wt}u(t),e^{-wt}p(t))\,\mathrm{d}t$ is anti-selfdual on $L^2_H$.*
2. *If $\ell$ is a selfdual boundary Lagrangian on $H\times H$, then the Lagrangian*
$$M(u,p)=\begin{cases}\int_0^T L(t,u(t),p(t)+\dot u(t))\,\mathrm{d}t+\ell(u(0),u(T)) & \text{if } u\in A^2_H,\\ +\infty & \text{otherwise}\end{cases}$$
*is anti-selfdual on $L^2_H$, provided $u\to\int_0^T L(t,u(t),p(t))\,\mathrm{d}t$ is continuous on $L^2_H$.*
3. *The Lagrangian defined on $A^2_H\times(A^2_H)^*$ by*
$$N(u,p)=\int_0^T L\left(t,u(t)-\int_t^T p(s)\,\mathrm{d}s,\dot u(t)\right)\mathrm{d}t+\ell\left(u(0)+\int_0^T p(s)\,\mathrm{d}s,u(T)\right)$$
*is anti-selfdual on $A^2_H\times X_0^*$ where $X_0^*=\{q\in(A^2_H)^*;\int_0^T q(s)\,\mathrm{d}s=0\}$.*

**Theorem 1.3.** *Suppose $L$ is an anti-selfdual Lagrangian on $[0,T]\times H\times H$ and $\ell$ is a selfdual boundary Lagrangian on $H\times H$, and suppose there exists $C>0$ such that for all $x\in L^2_H$, $\int_0^T L(t,x(t),0)\,\mathrm{d}t\leqslant C(1+\|x\|^2_{L^2_H})$. Then, there exists $v\in A^2_H$ such that*
$$\int_0^T L\big(t,v(t),\dot v(t)\big)\,\mathrm{d}t+\ell\big(v(0),v(T)\big)=\inf_{u\in A^2_H}\int_0^T L\big(t,u(t),\dot u(t)\big)\,\mathrm{d}t+\ell\big(u(0),u(T)\big)=0.$$

*In particular, for every $v_0 \in H$ the following functional on $A_H^2$*

$$I(u) = \int_0^T L\big(t, u(t), \dot{u}(t)\big)\,\mathrm{d}t + \frac{1}{2}\big\|u(0)\big\|^2 - 2\big\langle v_0, u(0)\big\rangle + \|v_0\|^2 + \frac{1}{2}\big\|u(T)\big\|^2$$

*has a minimum equal to zero. It is attained at a unique path $v$ which then satisfies the following*:

$$v(0) = v_0 \quad \textit{and} \quad \big(v(t), \dot{v}(t)\big) \in \mathrm{Dom}(L) \quad \textit{for almost all } t \in [0, T], \tag{1}$$

$$\big(-\dot{v}(t), -v(t)\big) \in \partial L\big(t, v(t), \dot{v}(t)\big), \tag{2}$$

$$\big\|v(t)\big\|_H^2 = \|v_0\|^2 - 2\int_0^t L\big(s, v(s), \dot{v}(s)\big)\,\mathrm{d}s \quad \textit{for every } t \in [0, T]. \tag{3}$$

Details and extensions are in [7,8] and [11].

## 2. Semigroups associated to autonomous anti-selfdual Lagrangians

When the Lagrangian $L(x, p)$ is autonomous, we can associate a flow without stringent boundedness or coercivity conditions. Indeed, we then use a Yosida-type $\lambda$-regularization of ASD-Lagrangians $L_\lambda = L \star T_\lambda$ where $T_\lambda(x, p) = \frac{\|x\|^2}{2\lambda^2} + \frac{\lambda^2\|p\|^2}{2}$. Then $L_\lambda$ satisfies the conditions of Theorem 1.3, and we can then find for each initial point $v \in H$, a path $v_\lambda \in A_H^2$, with $v_\lambda(0) = v$, which verify properties (1)–(3). Letting $\lambda \to 0$, we can recover a semi-group of 1-Lipschitz operators $T_t$ defined on the *Partial Domain* of $\partial L$ defined as $\mathrm{Dom}_1(\partial L) = \{x \in X;\ \text{there exists } p, q \in X^* \text{ such that } (p, 0) \in \partial L(x, q)\}$.

**Theorem 2.1.** *Let $L$ be an anti-selfdual Lagrangian on a Hilbert space $H$ that is uniformly convex in the first variable. Assuming* $\mathrm{Dom}_1(\partial L)$ *is non-empty, then there exists a semi-group of* 1*-Lipschitz operators* $(T_t)_{t\in\mathbf{R}^+}$ *on* $\mathrm{Dom}_1(\partial L)$ *such that* $T_0 = \mathrm{Id}$ *and for any* $x \in \mathrm{Dom}_1(\partial L)$*, the path* $x(t) = T_t x$ *satisfies* (2) *and* (3)

The following was established in [10] in the case of gradient flows of convex potentials (i.e., when $A = 0$ and $\omega = 0$), and in [9] in the case of gradient flows of semi-convex functions (i.e., when $A = 0$ and $\omega > 0$).

**Theorem 2.2.** *Let $\varphi$ be a proper, bounded below, convex lower semi-continuous functional on $H$ such that* $0 \in \mathrm{Dom}\,\partial\varphi$ *and let $A$ be a positive bounded linear operator on $H$. For any $\omega \in \mathbf{R}$ and $v_0 \in \mathrm{Dom}\,\partial\varphi$, consider the following functional on $A_H^2$*:

$$I(u) = \int_0^T \mathrm{e}^{-2\omega t}\big\{\psi\big(\mathrm{e}^{\omega t}u(t)\big) + \psi^*\big(\mathrm{e}^{\omega t}\big(-A^a u(t) - \dot{u}(t)\big\}\,\mathrm{d}t + \frac{1}{2}\big\|u(0)\big\|^2 - 2\big\langle v_0, u(0)\big\rangle + \|v_0\|^2 + \frac{1}{2}\big\|u(T)\big\|^2$$

*where $A^a$ is the anti-symmetric part of $A$, and $\psi(u) = \varphi(u) + \frac{1}{2}\langle Au, u\rangle$. The minimum of $I$ is then zero and is attained at a path $x(t)$, in such a way that $v(t) = \mathrm{e}^{\omega t}x(t)$ is a solution of*

$$\begin{cases} -Av(t) + \omega v(t) - \dot{v}(t) \in \partial\varphi\big(v(t)\big) & \textit{a.e. } t \in [0, T], \\ v(0) = v_0. \end{cases} \tag{4}$$

**Proposition 2.1.** *Let $\varphi$ be a proper convex lower semi-continuous function on $X \times Y$ and let $A : X \to Y^*$ be any bounded linear operator. Assume $B_1 : X \to X$ (resp., $B_2 : Y \to Y$) are positive operators, then for any $(x_0, y_0) \in$* $\mathrm{dom}(\partial\varphi)$ *and any $(f, g) \in X \times Y$, there exists a path $(x(t), y(t)) \in A_X^2 \times A_Y^2$ such that*

$$\begin{aligned}
&-\dot{x}(t) - A^* y(t) - B_1 x(t) + f \in \partial_1 \varphi\big(x(t), y(t)\big),\\
&-\dot{y}(t) + A y(t) - B_2 y(t) + g \in \partial_2 \varphi\big(x(t), y(t)\big),\\
&x(0) = x_0 \quad \textit{and} \quad y(0) = y_0.
\end{aligned}$$

*The solution is obtained as a minimizer on $A_X^2 \times A_Y^2$ of the following functional*

$$\begin{aligned}
I(x,y) = {}&\int_0^T \big\{\psi\big(x(t), y(t)\big) + \psi^*\big(-A^* y(t) - B_1^a x(t) - \dot{x}(t), A x(t) - B_2^a y(t) - \dot{y}(t)\big)\big\}\,\mathrm{d}t\\
&+ \frac{1}{2}\big\|x(0)\big\|^2 - 2\big\langle x_0, x(0)\big\rangle + \|x_0\|^2 + \frac{1}{2}\big\|x(T)\big\|^2 + \frac{1}{2}\big\|y(0)\big\|^2 - 2\big\langle y_0, y(0)\big\rangle + \|y_0\|^2 + \frac{1}{2}\big\|y(T)\big\|^2
\end{aligned}$$

*whose infimum is zero. Here $B_1^a$ (resp., $B_2^a$) are the skew-symmetric parts of $B_1$ and $B_2$ and $\psi(x,y) = \varphi(x,y) + \frac{1}{2}\langle B_1 x, x\rangle - \langle f, x\rangle + \frac{1}{2}\langle B_2 y, y\rangle - \langle g, x\rangle$.*

## 3. Variational resolution for general parabolic equations

For $t \in [0,T]$, consider $(b_1^t, b_2^t): X_t \to H_1^t \times H_2^t$ to be *regular boundary operators* from a reflexive Banach space $X_t$ into Hilbert spaces $H_1^t, H_2^t$ and we let $\Lambda_t : X_t \to X_t^*$ be *skew-adjoint operators modulo the boundary* $(b_1^t, b_2^t)$: that is for every $x, y \in X_t$, we have $\langle \Lambda_t x, y\rangle_H = -\langle \Lambda_t y, x\rangle_H + \langle b_2^t(x), b_2^t(y)\rangle_{H_2^t} - \langle b_1^t(x), b_1^t(y)\rangle_{H_1^t}$. We refer to [5,7] for the details. Suppose now $H$ is a Hilbert space such that:

$$X_t \subset H \subset X_t^* \text{ is an evolution triple with } \operatorname{Ker}(b_1^t, b_2^t) \text{ being dense in } H. \tag{5}$$

$$\Lambda_t(X_t) \subset H \text{ and } X_t = \big\{x \in H; \sup\big\{\langle x, \Lambda_t y\rangle_H;\ y \in X_t,\ \|y\|_H \leqslant 1\big\} < +\infty\big\}. \tag{6}$$

We then call $(X_t, H, \Lambda_t)$ a *maximal evolution triple*. Starting now with a time-dependent ASD Lagrangian $L$ on $H$, and selfdual *state-boundary Lagrangians* $m_t : H_1^t \times H_2^t \to \mathbb{R} \cup \{+\infty\}$ one can prove (see [7]) that

$$M(t,x,p) = \begin{cases} L(t, x, \Lambda_t x + p) + m_t\big(b_1^t(x), b_2^t(x)\big) & \text{if } x \in X_t,\\ +\infty & \text{otherwise} \end{cases}$$

is also anti-selfdual on $H \times H$ for each $t \in [0,T]$.

If now $\ell$ is a selfdual time-boundary Lagrangian on $H$, then the following Lagrangian $\widetilde{M}(u,p) = \int_0^T \{M(t, u(t), p(t) + \dot{u}(t))\}\,\mathrm{d}t + \ell(u(0), u(T))$ is anti-selfdual Lagrangian on the elements of $A_H^2 \times \{0\}$ which is sufficient to get that $I(u) = \widetilde{M}(u,0) = \int_0^T \{L(t, u(t), \Lambda_t u(t) + \dot{u}(t)) + m(t, b_1^t u(t), b_2^t u(t))\}\,\mathrm{d}t + \ell(u(0), u(T))$ has a minimum at $\bar{v}(t)$, and that the minimal value is zero. Applying the theorem with the time boundary Lagrangian on $H$, $\ell(x,p) = \frac{1}{2}\|x\|^2 - 2\langle a, x\rangle + \|a\|^2 + \frac{1}{2}\|p\|^2$, where $a$ is a given initial value in $H$, and with a state boundary Lagrangian $m(t,x,p) = \frac{1}{2}\|x\|^2 - 2\langle b(t), x\rangle + \|b(t)\|^2 + \frac{1}{2}\|p\|^2$, where $b(t)$ is prescribed in $H_1^t$ for each $t$, we get that $\bar{v}(t)$ satisfies:

$$\begin{cases} L\big(t, v(t), \Lambda_t v(t) + \dot{v}(t)\big) + \big\langle v(t), \Lambda_t v(t) + \dot{v}(t)\big\rangle = 0 \quad \text{a.e. } t \in [0,T],\\ \big(-\Lambda_t v(t) - \dot{v}(t), -v(t)\big) \in \partial L\big(t, v(t), \dot{v}(t)\big), b_1^t\big(v(t)\big) = b(t) \quad \text{a.e. } t \in [0,T],\\ v(0) = a. \end{cases} \tag{7}$$

**Theorem 3.1.** *Under the above conditions on $(X_t, H, H_1^t, H_2^t, b_1^t, b_2^t)$, consider bounded linear operators $A_t : X_t \to X_t^*$ such that $A_t - \frac{1}{2}((b_2^t)^* b_2^t - (b_1^t)^* b_1^t)$ is positive and denote by $\Lambda_t$ the operator $\Lambda_t = \frac{1}{2}(A_t - A_t^*) + \frac{1}{2}((b_2^t)^* b_2^t - (b_1^t)^* b_1^t)$ which is skew-adjoint modulo the boundary. For each $t \in [0,T]$, suppose $(X_t, H, \Lambda_t)$ is a* *maximal evolution triple and that $\varphi(t,\cdot)$ is a convex continuous function on $H$. For $f \in L^2([0,T];H)$, $a \in H$ and $b(t) \in H_t^1$ consider the following functional on $A_H^2$,*

$$I(u) = \int_0^T \Big\{ \psi\big(t, u(t)\big) + \psi^*\big(t, -\Lambda_t u(t) - \dot u(t)\big) + \frac{1}{2}\big(\big\|b_1^t u(t)\big\|^2 + \big\|b_2^t u(t)\big\|^2\big) - 2\big\langle b(t), u(t)\big\rangle + \big\|b(t)\big\|^2 \Big\}\,\mathrm{d}t$$
$$+ \frac{1}{2}\big(\big\|u(0)\big\|^2 + \big\|u(T)\big\|^2\big) - 2\big\langle u(0), a\big\rangle + \|a\|^2,$$

*where $\psi(t,x) = \varphi(t,x) + \frac{1}{2}\langle A_t x, x\rangle - \frac{1}{4}(\|b_2^t x\|^2 - \|b_1^t x\|^2) + \langle f(t), x\rangle$. Suppose there is $C > 0$ so that for every $x \in L_H^2$, $\int_0^T \psi(t, x(t)) + \psi^*(t, -\Lambda_t x(t))\,\mathrm{d}t \leqslant C(1 + \|x\|^2_{L_H^2})$.*

*Then there exists $v \in A_H^2$ such that $I(v) = \inf_{u \in A_H^2} I(u) = 0$. Moreover, $v$ solves*

$$\begin{cases} -A_t v(t) - \dot v(t) \in \partial\varphi\big(t, v(t)\big) + f(t) & \text{a.e. } t \in [0,T], \\ b_1^t(v(t)) = b(t) & \text{a.e. } t \in [0,T], \\ v(0) = a. \end{cases} \tag{8}$$

### 3.1. Non-linear transport evolutions

With the hypothesis of [2], we consider the equation

$$\begin{cases} -\frac{\partial u}{\partial t} - \sum_{i=1}^n a_i \frac{\partial u}{\partial x_i} - a_0 u = \beta(u) + f & \text{on } [0,T] \times \Omega, \\ u(t,x) = b(t,x) & \text{on } [0,T] \times \Sigma_-, \\ u(0,x) = u_0(x) & \text{on } \Omega \end{cases} \tag{9}$$

where $u_0 \in H_A^1(\Omega)$, $f \in H_A^1(\Omega)^*$ and where $b(t) \in L^2(\Sigma, \mathbf{n}\cdot\mathbf{a}\,\mathrm{d}x)$ for each $t \in [0,T]$. Let $\psi(u) = \int_\Omega \{j(u(x)) + f(x)u(x) + \frac{1}{2}(a_0 - \frac{1}{2}\operatorname{div} a)u^2\}\,\mathrm{d}x$.

**Theorem 3.2.** *Assume $a_0(x) - \frac{1}{2}\operatorname{div} a(x) \geqslant \alpha > 0$ on $\Omega$, and consider the following functional on the space $X := A^2([0,T]; H_A^1(\Omega))$.*

$$I(u) = \int_0^T \Big\{ \psi\big(u(t)\big) + \psi^*\Big(-\mathbf{a}\cdot\nabla_x u(t) - \frac{1}{2}\operatorname{div} a u(t) - \dot u(t)\Big)\Big\}\,\mathrm{d}t$$
$$+ \int_0^T \Big\{ \frac{1}{2}\int_{\Sigma_+} \big|u(t,x)\big|^2 \mathbf{n}\cdot\mathbf{a}\,\mathrm{d}\sigma - \frac{1}{2}\int_{\Sigma_-} \big|u(t,x)\big|^2 \mathbf{n}\cdot\mathbf{a}\,\mathrm{d}\sigma + \int_{\Sigma_-} \big(\big|b(t,x)\big|^2 - 2b(t,x)u(t,x)\big)\mathbf{n}\cdot\mathbf{a}\,\mathrm{d}\sigma \Big\}\,\mathrm{d}t$$
$$+ \int_\Omega \Big\{ \frac{1}{2}\big(\big|u(0,x)\big|^2 + \big|u(x,T)\big|^2\big) - 2\big\langle u(0,x), u_0(x)\big\rangle + \big|u_0(x)\big|^2 \Big\}\,\mathrm{d}x.$$

*There exists then $\bar u \in X$ such that $I(\bar u) = \inf_{u \in X} I(u) = 0$ and which solves Eq.* (9).

## 4. Variational resolution for parabolic variational inequalities

Consider for each time $t$, a bilinear continuous functional $a_t$ on a Hilbert space $H \times H$ and a convex l.s.c. function $\varphi(t,\cdot): H \to \mathbf{R} \cup \{+\infty\}$. Solving the corresponding parabolic variational inequality amounts to constructing for a given $f \in L^2([0,T];H)$ and $x_0 \in H$, a path $x(t) \in A_H^2([0,T])$ such that for all $z \in H$,

$$\big\langle \dot{x}(t), x(t) - z\big\rangle + a_t\big(x(t), x(t) - z\big) + \varphi\big(t, x(t)\big) - \varphi(t,z) \leqslant \big\langle x(t) - z, f(t)\big\rangle \tag{10}$$

for almost all $t \in [0,T]$. This problem can be rewritten as: $f(t) \in \dot{y}(t) + A_t y(t) + \partial\varphi(t,y)$, where $A_t$ is the bounded linear operator on $H$ defined by $a_t(u,v) = \langle A_t u, v\rangle$. This means that the variational inequality (10) can be rewritten and solved using the variational principle in Theorem 3.1. For example, under appropriate conditions one can then solve variationally the following "obstacle" problem.

If $K$ is a convex closed subset of $H$, then for any $f \in L^2([0,T];H)$ and any $x_0 \in K$, there exists a path $x \in A_H^2([0,T])$ such that $x(0) = x_0$, $x(t) \in K$ for almost all $t \in [0,T]$ and $\langle \dot{x}(t), x(t) - z\rangle + a_t(x(t), x(t) - z) \leqslant \langle x(t) - z, f\rangle$ for all $z \in K$. The path $x(t)$ is obtained as a minimizer of the following functional on $A_H^2([0,T])$:

$$I(y) = \int_0^T \big\{\varphi\big(t, y(t)\big) + \big(\varphi(t,\cdot) + \psi_K\big)^*\big(-\dot{y}(t) - \Lambda_t y(t)\big)\big\}\,\mathrm{d}t + \frac{1}{2}\big(\big|y(0)\big|^2 + \big|y(T)\big|^2\big) - 2\big\langle y(0), x_0\big\rangle + |x_0|^2.$$

Here $\varphi(t,y) = \frac{1}{2}a_t(y,y) - \langle f(t), y\rangle$, $\Lambda_t : H \to H$ is the skew-adjoint operator defined by $\langle \Lambda_t u, v\rangle = \frac{1}{2}(a_t(u,v) - a_t(v,u))$, and $\psi_K(y) = 0$ on $K$ and $+\infty$ elsewhere.

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