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Algebraic Geometry

A lower bound for the dimension of the base locus of the generalized theta divisor

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Abstract

We produce a lower bound for the dimension of the base locus of the generalized theta divisor Θ_r on the moduli space $\mathcal{SU}_C(r)$ of semistable vector bundles of rank r and trivial determinant on a smooth curve C of genus $g \geqslant 2$. ***To cite this article: D. Arcara, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Une borne inférieure pour la dimension du lieu de base du diviseur thêta généralisé. Nous déterminons une borne inférieure pour la dimension du lieu de base du diviseur thêta généralisé Θ_r sur l'espace des modules $\mathcal{SU}_C(r)$ des fibrés vectoriels semi-stables de rang r et de determinant trivial sur une courbe lisse C de genre $g \geqslant 2$. ***Pour citer cet article : D. Arcara, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

1. Introduction

Let C be a smooth irreducible complex projective curve of genus $g \geqslant 2$. Let $\mathcal{U}_C(r)$ be the moduli space of (S-equivalence classes of) semi-stable vector bundles of rank r and degree 0, and let $\mathcal{SU}_C(r)$ be the moduli space of (S-equivalence classes of) semi-stable vector bundles of rank r and trivial determinant.

The Picard group of $\mathcal{SU}_C(r)$ is generated by an ample line bundle (see [4]), that we shall denote by $\mathcal{L}_r$. A divisor Θ_r on $\mathcal{SU}_C(r)$ such that $\mathcal{L}_r = \mathcal{O}_{\mathcal{SU}_C(r)}(\Theta_r)$ is called a generalized theta divisor. We are interested in the base locus of its linear system.

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A vector bundle $E \in \mathcal{SU}_C(r)$ is in the base locus of the generalized theta divisor if and only if $H^0(E \otimes L) \neq 0$ for every line bundle L on C of degree $g-1$ (see [1] for $r=2$ and [3] in general). Raynaud studied bundles with a similar property in [8], and Beauville summarizes his results as follows in [2].

Theorem 1.1 (Raynaud).

(a) *For $r=2$, the linear system $|\Theta_2|$ has no base points.*
(b) *For $r=3$, $|\Theta_3|$ has no base points if $g=2$, or if $g \geqslant 3$ and C is generic.*
(c) *Let n be an integer $\geqslant 2$ dividing g. For $r=n^g$, the linear system $|\Theta_r|$ has base points.*

For part (c), Raynaud actually constructs, for every $n \geqslant 2$ and g, a vector bundles of rank n^g and slope g/n without the property that he calls $(*)$. The bundles without the property $(*)$ can be easily used to produce vector bundles in the base locus of the generalized theta divisor if their slope is integral, hence part (c) of the theorem above. Note also that Popa generalized Raynaud's construction in [7].

Using the dual E_L of the kernel of the evaluation map e_L for a line bundle L generated by its global sections, and its exterior powers, Popa [6] and Schneider [9] proved the existence of other vector bundles in the base locus of the generalized theta divisor.

In particular, Schneider defines a condition (R), which implies Raynaud's condition $(*)$, as follows: A vector bundles E has the property (R) if, for every $n \in \mathbb{Z}$ and any generic line bundle L of degree n, $H^0(E \otimes L) = 0$ or $H^1(E \otimes L) = 0$. He then proves the following proposition.

Proposition 1.2 (Schneider). *Let C be a smooth complex projective curve of genus $g \geqslant 2$. If L is a line bundle of degree greater than or equal to $2g+2$, then $\Lambda^p E_L$ does not verify (R) for every $p \in \{2, \ldots, \mathrm{rk}(E_L) - 2\}$.*

Under the assumption of the proposition, E_L is stable (see [5]), and therefore $\Lambda^p E_L$ is semi-stable. Whenever the slope of $\Lambda^p E_L$ is integral, this easily produces examples of vector bundles in the base locus of the generalized theta divisor.

As our first result, we prove that every vector bundles without the property (R) 'produces' a vector bundle in the base locus of the generalized theta divisor, hence making it possible to use all of the bundles studied by Raynaud, Popa, and Schneider, even the ones with non-integral slope.

Theorem 1.3. *If E is a semi-stable vector bundle of rank r on C which does not satisfy the property (R), then the base locus of $|\Theta_r|$ is non-empty.*

As a corollary, using Raynaud's and Schneider's results, we obtain the following corollary.

Corollary 1.4. *The base locus of $|\Theta_r|$ is non-empty for $r = 2^g$ and $r = (g+1)(g+2)/2$.*

As Popa points out in [6], this implies that the base locus is also non-empty for any bigger rank (if E is in the base locus of $|\Theta_r|$, just take $E \oplus \mathcal{O}_C^{\oplus n}$). If we let r_0 be the lowest rank such that the base locus of $|\Theta_r|$ is non-empty, the corollary above can be restated as

$$r_0 \leqslant \min\left\{2^g, \frac{(g+1)(g+2)}{2}\right\}.$$

We produce the following lower bound for the dimension of the base locus of $|\Theta_r|$:

Proposition 1.5. *Let C be a smooth complex projective curve of genus g. Then the dimension of the base locus of $|\Theta_r|$ is at least $(r - r_0)^2(g-1) + 1$, where r_0 is the minimum rank for which the base locus of the generalized theta divisor is non-empty.*

2. Proof of Theorem 1.3

If μ is a rational number, we shall denote by $\lfloor\mu\rfloor$ the largest integer less than or equal to μ and by $\lceil\mu\rceil$ the smallest integer greater than or equal to μ.

Schneider proves in [9] that a vector bundle E satisfies the condition (R) if and only if it satisfies the two following conditions:

(1) $H^1(E\otimes L)=0$ for a generic line bundle L of degree $g-1-\lfloor\mu(E)\rfloor$;
(2) $H^0(E\otimes L)=0$ for a generic line bundle L of degree $g-1-\lceil\mu(E)\rceil$.

Let E be a semi-stable vector bundle of rank r which does not satisfy the property (R). If $\mu(E)$ is an integer, then $E\otimes L$ is in the base locus of $|\Theta_r|$, where L is a line bundle of degree $g-1-\mu(E)$ such that $L^{-r}\simeq\det E$. If $\mu(E)$ is not an integer, there are two cases.

Case I: For every line bundle L of degree $g-1-\lfloor\mu(E)\rfloor$, $H^1(E\otimes L)\neq 0$.

Let $\mathbb{C}_p$ be a skyscraper sheaf of degree 1 supported at a point p of C, let $E\to\mathbb{C}_p$ be a non-zero map, and let E' be the kernel: $0\to E'\to E\to\mathbb{C}_p\to 0$. Since $H^1(\mathbb{C}_p)=0$, E' also satisfies the condition that $H^1(E'\otimes L)\neq 0$ for every line bundle L of degree $g-1-\lfloor\mu(E)\rfloor$. Moreover, $\lfloor\mu(E')\rfloor=\lfloor\mu(E)\rfloor$. There are now two subcases.

Subcase I.1: E' is semi-stable. Then E' is a semi-stable vector bundle of slope $\mu(E')<\mu(E)$ with $\lfloor\mu(E')\rfloor=\lfloor\mu(E)\rfloor$ such that $H^1(E'\otimes L)\neq 0$ for every line bundle L of degree $g-1-\lfloor\mu(E')\rfloor$.

Subcase I.2: E' is not semi-stable. Let μ be the maximum slope of a vector sub-bundle of E', and let F' be a sub-bundle of maximal rank among all of the sub-bundles of slope μ. Then there exists a short exact sequence $0\to F'\to E'\to G'\to 0$ with F' and G' stable vector bundles. By semi-continuity, we obtain that either $H^1(F'\otimes L)\neq 0$ for every line bundle L of degree $g-1-\lfloor\mu(E)\rfloor$ or $H^1(G'\otimes L)\neq 0$ for every line bundle L of degree $g-1-\lfloor\mu(E)\rfloor$. Let us show that $\mu(F')$ and $\mu(G')$ are both $\geqslant\lfloor\mu(E)\rfloor$. Clearly, $\mu(F')>\mu(E')\geqslant\lfloor\mu(E)\rfloor$. For G', note that it is contained in $G=E/F'$, and $\mu(G)\geqslant\mu(E)>\lfloor\mu(E)\rfloor$. Therefore, $\mu(G')=\mu(G)-1/\mathrm{rk}(G)\geqslant\lfloor\mu(E)\rfloor$. Since $\mu(F')$ and $\mu(G')$ are clearly $\leqslant\mu(E)$, this shows that $\lfloor\mu(F')\rfloor=\lfloor\mu(G')\rfloor=\lfloor\mu(E)\rfloor$. Therefore, either F' or G' is a semi-stable vector bundle E'' of slope $\mu(E'')\geqslant\lfloor\mu(E)\rfloor$ and rank $\mathrm{rk}(E'')<r$ with $\lfloor\mu(E'')\rfloor=\lfloor\mu(E)\rfloor$ such that $H^1(E''\otimes L)\neq 0$ for every line bundle L of degree $g-1-\lfloor\mu(E'')\rfloor$.

We can now continue our process by replacing E with either E', if it is semi-stable, or with the E'' constructed in the case when E' is not semi-stable. The process eventually ends when the slope becomes integral. This happens because the slopes of the vector bundles constructed at each step is bounded below by $\lfloor\mu(E)\rfloor$ and at each step either the slope or the rank is decreasing.

To conclude the proof in this case, note that, if the vector bundle in the base locus of the generalized theta divisor constructed has rank $r'<r$, we can always produce one in the base locus of the generalized theta divisor by taking its direct sum with $r-r'$ copies of $\mathcal{O}_C$.

Case II: For every line bundle L of degree $g-1-\lceil\mu(E)\rceil$, $H^0(E\otimes L)\neq 0$.

The proof in this case is very similar to the proof of Case I, except that we now start by taking E' to be a non-trivial extension of $\mathbb{C}_p$ by E.

3. Proof of Proposition 1.5

To simplify the proof of Proposition 1.5, let us point out the following result, which is probably well-known. We prove it here because we could not find a proof in the literature.

Proposition 3.1. *Every vector bundle in the base locus of $|\Theta_{r_0}|$ is stable.*

Proof. Let E be a vector bundle in the base locus of $|\Theta_{r_0}|$. Then E is in the same equivalence class as its associated grading $\bigoplus_{i=1}^k \mathrm{Gr}_i$ from its Jordan–Hölder filtration. Since $H^0(E \otimes L) \neq 0$ if and only if there exists an i such that $H^0(\mathrm{Gr}_i \otimes L) \neq 0$, by semicontinuity there exists an i such that Gr_i is in the base locus of $|\Theta_{\mathrm{rk}\,\mathrm{Gr}_i}|$. By the minimality of r_0, $\mathrm{rk}\,\mathrm{Gr}_i = r_0$, $k = 1$, and $E = \mathrm{Gr}_1$ is stable. □

We are now ready to prove Proposition 1.5. Let E be a vector bundle in the base locus of $|\Theta_{r_0}|$. Then E is stable by Proposition 3.1. Let $n = r - r_0$, and consider the morphism

$$\varphi : JC \times \mathcal{U}_C(n) \to \mathcal{U}_C(r), \quad \varphi(L, F) = (E \otimes L) \oplus F,$$

where JC is the Jacobian of C. If we let $A = \{(L, F) \in JC \times \mathcal{U}_C(n) \mid L^r \otimes \det F \simeq \mathcal{O}_C\}$, then, by projecting A onto $\mathcal{U}_C(n)$, it is easy to see that $\dim A = \dim \mathcal{U}_C(n) = n^2(g-1) + 1$. Moreover, $\dim \varphi(A) = \dim A$, and $\varphi(A) = \varphi(JC \times \mathcal{U}_C(n)) \cap \mathcal{SU}_C(r)$ is contained in the base locus of $|\Theta_r|$.

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