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Dynamical Systems/Ordinary Differential Equations

A Note on non-autonomous scalar functional differential equations with small delay

Ana I. Alonso [a], Rafael Obaya [a,1], Ana M. Sanz [b]

[a] *Departamento de Matemática Aplicada, E.T.S. de Ingenieros Industriales, Paseo del Cauce s/n, 47011 Valladolid, Spain*
[b] *Departamento de Análisis Matemático y Didáctica de la Matemática, Facultad de Ciencias, Prado de la Magdalena s/n, 47005 Valladolid, Spain*

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Abstract

We prove that the minimal sets in the skew-product semiflows generated from a non-autonomous scalar functional differential equation with a small delay are all almost automorphic extensions of the base. This result is not true for arbitrary delay equations. The point is that, for a small delay, so-called *special solutions* exist and permit us to tackle the problem by means of some related scalar ODE's for which the study is much simpler. ***To cite this article: A.I. Alonso et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Une Note sur les équations fonctionnelles scalaires non-autonomes à petit retard. Dans cette Note on montre que les ensembles minimaux pour les semiflots engendrés par les solutions des équations fonctionnelles non-autonomes à petit retard sont des extensions presque automorphes de la base. Ce résultat n'est plus vrai pour un retard arbitraire. C'est la condition sur le retard qui garantit l'existence de solutions dites *solutions spéciales*. Ces solutions-ci nous permettent de considérer notre problème au moyen d'un autre plus facile relatif aux équations différentielles ordinaires. ***Pour citer cet article : A.I. Alonso et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

E-mail addresses: anaalo@wmatem.eis.uva.es (A.I. Alonso), rafoba@wmatem.eis.uva.es (R. Obaya), anasan@wmatem.eis.uva.es (A.M. Sanz).

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La dynamique presque automorphe (p.a.) s'est montrée essentielle dans l'étude des équations différentielles presque périodiques (p.p.). En effet, une variation temporelle p.p. des coefficients de l'équation peut engendrer un comportement p.a. mais non p.p. des solutions bornées ; on renvoie à Johnson [4] et Zhikov et Levitan [11] pour des exemples. À partir de quelques systèmes presque périodiques de Millionščikov et Vinograd, Johnson [5] construit des exemples où il existe un ensemble minimal p.a. qui n'est pas uniquement ergodique. En fait, la structure mesurable de tels ensembles p.a. peut être très compliquée dans certains cas (on renvoie aussi à Furstenberg et Weiss [3]). Récemment Novo et al. [8] ont montré que dans le contexte des équations différentielles ordinaires scalaires convexes la présence d'un ensemble minimal p.a. mais pas p.p. est rare, dans un sens topologique précis.

Dans le cas de la dynamique skew-product unidimensionnelle, les ensembles minimaux sont des extensions presque automorphes du flot sur la base (Theorem 3.1). Dans le cas de la dimension infinie, pour les semiflots skew-product engendrés par des équations scalaires paraboliques avec 1-dimension spatiale, Shen et Yi [10] ont prouvé le même résultat. Cependant, ce résultat n'est pas vrai pour des équations scalaires générales à retard : selon Krisztin et al. [6], il y a des équations scalaires autonomes à retard qui ont trois points fixes, dont toutes les solutions sont bornées, et qui admettent de plus une solution périodique non triviale (dans le cas autonome, un ensemble minimal extension p.a. de la base est ponctuel). Le but de cette Note est de montrer que, lorsque le retard est suffisamment petit, le résultat est encore vrai.

La première section commence avec quelques définitions et on y construit, en partant d'une équation récurrente à retard du type $y' = f(t, y_t)$, une famille d'Éqs. (2) (cf. version anglaise)

$$y' = F(\omega \cdot t, y_t), \quad \omega \in \Omega,$$

indexée sur l'enveloppe minimale Ω, dont les solutions engendrent un semiflot skew-product (3) sur l'espace produit $\Omega \times X$, où $X = C([-r, 0], \mathbb{R})$. Dans la seconde section, sous l'hypothèse de petit retard (iii), on trouve l'existence de solutions spéciales de (2) qui sont définies sur toute la droite $\mathbb{R}$ et qui sont en même temps les solutions d'un problème d'équations différentielles ordinaires (7) intimement lié au problème initial. Concrètement, on peut énoncer le résultat suivant (pour la première part, on renvoie à Driver [2]).

Théorème 0.1. *Si la fonction F* (i) *est continue sur $\Omega \times X$ et* (ii) *satisfait la condition de Lipschitz $|F(\omega, v_1) - F(\omega, v_2)| \leqslant L \|v_1 - v_2\|$ pour tout $\omega \in \Omega$ et $v_1, v_2 \in X$, et de plus* (iii) *$L \cdot r \cdot e < 1$, alors pour chaque $\omega \in \Omega$, $t_0 \in \mathbb{R}$ et $x_0 \in \mathbb{R}$ il existe une unique solution de* (2) *définie sur tout $\mathbb{R}$, $y(\omega, t_0, x_0)(t)$, $t \in \mathbb{R}$, qu'on appelle solution spéciale, telle que*

$$y(\omega, t_0, x_0)(t_0) = x_0 \quad \textit{et} \quad \sup_{t \leqslant t_0} \left| y(\omega, t_0, x_0)(t) \right| \mathrm{e}^{t/r} < \infty.$$

Par ailleurs, pour chaque $\omega \in \Omega$ et $t_0, x_0 \in \mathbb{R}$ la solution spéciale $y(\omega, t_0, x_0)$ de (2) *est précisément l'unique solution du problème de Cauchy* (7).

Finalement, on remarque que dans un ensemble minimal $M \subset \Omega \times X$ toutes les orbites admettent une prolongement aux temps négatifs, bien qu'éventuellement non unique. Alors, si on considère une orbite complète dans M, la solution correspondante de (2) est définie sur tout $\mathbb{R}$ et elle est bornée, donc c'est une solution spéciale. Ainsi, dans la dernière section on donne le résultat clef de cette Note :

Théorème 0.2. *On fait les hypothèses* (i) *et* (ii) *sur la fonction F de* (2), *et l'hypothèse* (iii) *sur le retard. Alors, si $M \subset \Omega \times X$ est un ensemble minimal pour le semiflot* (3), *M est une extension presque automorphe du flot sur la base (Ω, σ). En particulier, quand le flot sur la base est presque périodique, M est un minimal presque automorphe.*

1. Introduction and preliminaries

Almost automorphic (a.a. for short) dynamics has proved to be fundamental in the study of almost periodic (a.p. for short) differential equations. The reason is that an a.p. time variation in the coefficients of the equation may produce an a.a. but not a.p. behavior of bounded solutions; see Johnson [4] and Zhikov and Levitan [11] for examples. Also Johnson, starting from some a.p. systems of Millionščikov and Vinograd, gives in [5] examples where there appears an a.a. minimal set which is not uniquely ergodic. Actually, it is well-known that the measurable structure of such an a.a. minimal set can exhibit high complexity in some cases (see Furstenberg and Weiss [3]). Recently Novo et al. [8] have shown that in the context of scalar convex ODE's the presence of an a.a. minimal set which is not a.p. is rather infrequent in a precise topological sense.

In abstract one-dimensional skew-product dynamics over a minimal base flow, and in particular in the skew-product flows generated by recurrent non-autonomous scalar ODE's, minimal sets are a.a. extensions of the base flow (see Theorem 3.1). In infinite dimensional dynamics, Shen and Yi [10] have proved that the same result holds for the minimal sets of the skew-product semiflow induced by scalar parabolic equations in 1-space dimension. However, it is no longer true for general scalar delay equations: according to Krisztin et al. [6], autonomous scalar quasimonotone delay differential equations having three equilibria and all solutions bounded may have a nontrivial periodic solution (notice that in the autonomous case minimal sets which are a.a. extensions of the base are singletons). It is the aim of this Note to prove that the result is true provided that the delay is small enough.

We begin with some definitions and preliminary results. Let (Ω,σ) be a continuous flow on a compact metric space. A compact invariant set $M\subseteq\Omega$ is *minimal* if it contains no non-empty proper closed invariant subset. A *flow homomorphism* from another continuous flow (Y,Ψ) to (Ω,σ) is a continuous map $\pi: Y\to\Omega$ such that $\pi(\Psi(t,y))=\sigma(t,\pi(y))$ for every $y\in Y$ and $t\in\mathbb{R}$. If π is also bijective, it is called a *flow isomorphism*. Let $\pi:(Y,\Psi)\to(\Omega,\sigma)$ be a surjective flow homomorphism and suppose (Y,Ψ) is minimal (then, so is (Ω,σ)). (Y,Ψ) is said to be an *almost automorphic extension* of (Ω,σ) if there is $\omega\in\Omega$ such that $\mathrm{card}(\pi^{-1}(\omega))=1$. Then, actually $\mathrm{card}(\pi^{-1}(\omega))=1$ for ω in a residual subset $\Omega_0\subseteq\Omega$; in the nontrivial case $\Omega_0\subsetneq\Omega$ the dynamics on Y can be very complicated. A minimal flow (Y,Ψ) is *almost automorphic* if it is an a.a. extension of an almost periodic minimal flow (Ω,σ). We refer the reader to the work of Shen and Yi [10] for a survey of almost periodic and almost automorphic dynamics.

Let $r>0$ be a real number. In what follows, we consider the Banach space $X=C([-r,0],\mathbb{R})$ with the norm $\|v\|=\sup_{t\in[-r,0]}|v(t)|$. A function $f\in C(\mathbb{R}\times X,\mathbb{R})$ is *admissible* if for any compact set $K\subset X$, f is bounded and uniformly continuous on $\mathbb{R}\times K$. We consider the scalar delay functional differential equation

$$y'=f(t,y_t),\tag{1}$$

where $f:\mathbb{R}\times X\to\mathbb{R}$ is an admissible function, and for each $t\geqslant 0$ the function $y_t\in X$ is defined as $y_t(s)=y(t+s)$ for each $s\in[-r,0]$. We assume in addition that for each bounded set $B\subset X$, f takes $\mathbb{R}\times B$ into a bounded set. Under these conditions, Ω, the *hull* of f, namely, the closure of the set of mappings $\{f_t\mid t\in\mathbb{R}\}$ ($f_t(s,v)=f(t+s,v)$) in the topology of uniform convergence on compact sets, is a compact metric space, and each $\omega\in\Omega$ is also an admissible function. Besides, the translation $\mathbb{R}\times\Omega\to\Omega$, $(t,\omega)\mapsto\omega\cdot t$ with $\omega\cdot t(s,v)=\omega(t+s,v)$ defines a continuous flow σ on Ω (see Novo et al. [7] for more details). The function f is said to be *recurrent* if it generates a minimal hull.

We will assume that (Ω,σ) is a minimal flow, which, among other cases, is satisfied when f is a uniformly a.p. or a uniformly a.a. function (see [7] for the definitions). It is well-known that $F:\Omega\times X\to\mathbb{R}$, $(\omega,v)\mapsto\omega(0,v)$ is the unique continuous extension of f to the hull, in the sense that for all $\omega\in\Omega$, $t\in\mathbb{R}$, $v\in X$, $F(\omega\cdot t,v)=\omega(t,v)$. F inherits, if any, the differentiability of f with respect to v. As a minimum f is supposed to be locally Lipschitzian in v with Lipschitz constant independent on t. Let us consider the family of delay equations over the hull (notice that, when $\omega=f$, we recover the initial equation (1))

$$y'=F(\omega\cdot t,y_t),\quad \omega\in\Omega.\tag{2}$$

By the standard theory of delay functional differential equations, for fixed $\omega \in \Omega$ and $v \in X$, Eq. (2) locally admits a unique solution $y(t,\omega,v)$ with initial value v, i.e., $y(t,\omega,v) = v(t)$ for each $t \in [-r,0]$. Therefore, (2) induces a local skew-product semiflow

$$\begin{aligned} \tau : \mathbb{R}^+ \times \Omega \times X &\to \Omega \times X, \\ (t,\omega,v) &\mapsto \big(\omega \cdot t, y(\omega,v)_t\big), \end{aligned} \tag{3}$$

where $y(\omega,v)_t(s) = y(t+s,\omega,v)$, for $s \in [-r,0]$, is a piece of the solution.

2. Special solutions of delay equations and related ODE's

Consider Eq. (1) where f is an admissible function satisfying a global Lipschitz assumption: $|f(t,v_1) - f(t,v_2)| \leqslant L\|v_1 - v_2\|$ for all $v_1, v_2 \in X$ and $t \in \mathbb{R}$, for certain constant $L > 0$. Then the function F in (2) fulfils the following conditions (which imply existence of solutions of (2) on all $[-r,\infty)$):

(i) $F : \Omega \times X \to \mathbb{R}$ is continuous;
(ii) $|F(\omega,v_1) - F(\omega,v_2)| \leqslant L\|v_1 - v_2\|$ for every $\omega \in \Omega$ and $v_1, v_2 \in X$.

Let us assume further the following smallness condition on the size of r:

(iii) $L \cdot r \cdot e < 1$.

For the proof of the following theorem we refer the reader to the work of Driver [2].

Theorem 2.1. *Assume that* (i)–(iii) *hold. Then for each* $\omega \in \Omega$, $t_0 \in \mathbb{R}$ *and* $x_0 \in \mathbb{R}$ *there exists a unique solution of* (2) *defined on the whole real line, which we denote by* $y(\omega,t_0,x_0)(t)$, $t \in \mathbb{R}$, *such that*

$$y(\omega,t_0,x_0)(t_0) = x_0 \quad \textit{and} \quad \sup_{t \leqslant t_0} \big|y(\omega,t_0,x_0)(t)\big| \, \mathrm{e}^{t/r} < \infty. \tag{4}$$

Moreover, if λ *is the unique solution of the characteristic equation*

$$\lambda = -L\,\mathrm{e}^{-\lambda r} \tag{5}$$

in the interval $(-1/r,0)$, *then for all* $\omega \in \Omega$, $t \leqslant t_0$, $x_1, x_2 \in \mathbb{R}$,

$$\big|y(\omega,t_0,x_1)(t) - y(\omega,t_0,x_2)(t)\big| \leqslant |x_1 - x_2|\,\mathrm{e}^{\lambda(t-t_0)}. \tag{6}$$

The solutions $y(\omega,t_0,x_0)(t)$, $t \in \mathbb{R}$, determined in the previous theorem are the so-called *special solutions*. We remark that in the previous theorem the fact that the equations are scalar is irrelevant, and the same holds for systems of equations.

Next, we intend to show how special solutions allow us to translate certain issues for scalar delay equations into the study of some related scalar ODE's. Our result is inspired in the paper by Pituk [9] for the autonomous case. Under assumptions (i)–(iii), for each $(\omega,x) \in \Omega \times \mathbb{R}$ we can define $G(\omega,x) = F(\omega, y(\omega,0,x)_0)$, where $y(\omega,0,x)_0$ is the restriction of the special solution to the interval $[-r,0]$. The function G is continuous on $\Omega \times \mathbb{R}$ and we can consider for fixed $\omega \in \Omega$ and $t_0, x_0 \in \mathbb{R}$ the problem

$$\begin{cases} x' = G(\omega \cdot t, x), \\ x(t_0) = x_0. \end{cases} \tag{7}$$

Theorem 2.2. *If conditions* (i)–(iii) *hold, then the function* G *is Lipschitzian in* x *uniformly on* Ω *with Lipschitz constant* $-\lambda$ (*see* (5)), *that is,*

$$\big|G(\omega,x_1)-G(\omega,x_2)\big|\leqslant-\lambda|x_1-x_2|\quad\textit{for every } \omega\in\Omega \textit{ and } x_1,x_2\in\mathbb{R}. \tag{8}$$

Furthermore, for each $\omega\in\Omega$ and $t_0,x_0\in\mathbb{R}$ the special solution $y(\omega,t_0,x_0)$ of (2) *coincides with the unique solution of the Cauchy problem* (7).

Proof. First of all, notice that by virtue of (6) for any $s\in[-r,0]$,

$$\big|y(\omega,0,x_1)_0(s)-y(\omega,0,x_2)_0(s)\big|=\big|y(\omega,0,x_1)(s)-y(\omega,0,x_2)(s)\big|\leqslant|x_1-x_2|\,\mathrm{e}^{-\lambda r}.$$

Then, (8) follows straightforward from (ii) and (5), and uniqueness of solutions for the initial value problem (7) is guaranteed.

We now consider the translated function $\tilde y(t)=y(\omega,t_0,x_0)(t+t_0)$, for which

$$\tilde y'(t)=F\big((\omega\cdot t_0)\cdot t,\tilde y_t\big)\quad\text{and}\quad\tilde y(0)=y(\omega,t_0,x_0)(t_0)=x_0.$$

Besides, $\sup_{t\leqslant0}|\tilde y(t)|\,\mathrm{e}^{t/r}=\sup_{t\leqslant t_0}|y(\omega,t_0,x_0)(t)|\,\mathrm{e}^{t/r}\,\mathrm{e}^{-t_0/r}<\infty$, so that by uniqueness of special solutions it must be $\tilde y=y(\omega\cdot t_0,0,x_0)$, that is, for every $t\in\mathbb{R}$, $y(\omega,t_0,x_0)(t+t_0)=y(\omega\cdot t_0,0,x_0)(t)$; in particular $y(\omega,t_0,x_0)_{t_0}=y(\omega\cdot t_0,0,x_0)_0$, and consequently, on the one hand, for every $\omega\in\Omega$, $t,x\in\mathbb{R}$,

$$F\big(\omega\cdot t,y(\omega,t,x)_t\big)=F\big(\omega\cdot t,y(\omega\cdot t,0,x)_0\big)=G(\omega\cdot t,x).$$

On the other hand, again by a uniqueness argument, it is immediate that

$$y\big(\omega,t,y(\omega,t_0,x_0)(t)\big)(s)=y(\omega,t_0,x_0)(s)\quad\text{for any } s\in\mathbb{R}.$$

Combining the last two equalities, we can conclude that for $x(t)=y(\omega,t_0,x_0)(t)$,

$$\begin{aligned}x'(t)&=F\big(\omega\cdot t,y(\omega,t_0,x_0)_t\big)=F\big(\omega\cdot t,y\big(\omega,t,y(\omega,t_0,x_0)(t)\big)_t\big)\\&=F\big(\omega\cdot t,y\big(\omega\cdot t,0,y(\omega,t_0,x_0)(t)\big)_0\big)=G\big(\omega\cdot t,y(\omega,t_0,x_0)(t)\big)=G\big(\omega\cdot t,x(t)\big).\end{aligned}$$

As in addition $x(t_0)=y(\omega,t_0,x_0)(t_0)=x_0$, the proof is finished. □

3. Minimal sets for scalar delay equations

We first state the above-mentioned result on minimal sets for one-dimensional skew-product flows.

Theorem 3.1. *If (Ω,σ) is a minimal flow and $M\subset\Omega\times\mathbb{R}$ is a minimal set under a continuous skew-product flow on $\Omega\times\mathbb{R}$, then M is an almost automorphic extension of (Ω,σ).*

Proof. We only give a sketch of the proof. Define for each $\omega\in\Omega$ the maps $x_1(\omega)=\inf\{x\in\mathbb{R}\mid(\omega,x)\in M\}$ and $x_2(\omega)=\sup\{x\in\mathbb{R}\mid(\omega,x)\in M\}$. These functions are respectively lower and upper semicontinuous on Ω. Thus, there exists a residual set $\Omega_0\subseteq\Omega$ whose elements are continuity points of both x_1 and x_2. One can prove that Ω_0 is also an invariant set. Consider $K_i=\mathrm{cls}\{(\omega,x_i(\omega))\mid\omega\in\Omega_0\}$, $i=1,2$, both closed and invariant subsets of M (cls stands for closure). Then $M=K_1=K_2$, by minimality of M. Finally, deduce that $x_1(\omega)=x_2(\omega)$ for all $\omega\in\Omega_0$; that is, for the natural projection homomorphism $\pi:M\to\Omega$, $(\omega,x)\mapsto\omega$, there is a unique pair in $M\cap\pi^{-1}(\omega)$ for all $\omega\in\Omega_0$, which completes the proof. □

We come back again to Eq. (1) and the induced family (2) over the hull. Recall that f is supposed to be an admissible function with minimal hull Ω and to satisfy a global Lipschitz assumption in v. Also the smallness condition (iii) in Section 2 is assumed. Our main result is the following.

Theorem 3.2. *Under the above conditions, any minimal set $M\subset\Omega\times X$ for the skew-product semiflow* (3) *induced by the solutions of* (2) *is an almost automorphic extension of the base flow (Ω,σ).*

Proof. To begin with, associate to (2) the family of ODE's $x' = G(\omega \cdot t, x)$, $\omega \in \Omega$ as detailed in Section 2, and denote by $\varsigma : \mathbb{R} \times \Omega \times \mathbb{R} \to \Omega \times \mathbb{R}$ the skew-product flow induced by their solutions. According to Theorem 2.2 we can write $\varsigma(t, \omega, x) = (\omega \cdot t, y(\omega, 0, x)(t))$ for any $(t, \omega, x) \in \mathbb{R} \times \Omega \times \mathbb{R}$, where $y(\omega, 0, x)(t)$ are the special solutions given in Theorem 2.1.

Secondly, notice that semiorbits in M are bounded and therefore the corresponding solutions of (2) exist in the large. We also know that in the minimal set M semiorbits always admit a backward extension, though, in principle, possibly not unique (see [10]). Now, if we take a full orbit in M, the implicitly given solution of (2) is defined on the whole $\mathbb{R}$ and it is bounded, so that in particular it satisfies the bound in (4) and thus, it is a special solution by Theorem 2.1.

To finish, consider the set $M_0 = \{(\omega, v(0)) \mid (\omega, v) \in M\} \subset \Omega \times \mathbb{R}$ and define the map $\Phi : M \to M_0$, $(\omega, v) \mapsto (\omega, v(0))$. We claim that Φ is an isomorphism of flows. By the construction the map is onto. The fact that it is injective follows from the uniqueness of special solutions, for if $(\omega, v_1), (\omega, v_2) \in M$ are such that $v_1(0) = v_2(0)$, then $y(\omega, 0, v_1(0)) \equiv y(\omega, 0, v_2(0))$, and as we have noticed in the previous paragraph, this implies in particular that $v_1 = v_2$ as elements of X. Finally, concerning the homomorphism property, for every $(\omega, v) \in M$ and $t \geqslant 0$,

$$\begin{aligned}\Phi\big(\tau(t, \omega, v)\big) &= \Phi\big(\omega \cdot t, y(\omega, v)_t\big) = \big(\omega \cdot t, y(\omega, v)_t(0)\big) \\ &= \big(\omega \cdot t, y(\omega, 0, v(0))(t)\big) = \varsigma\big(t, \omega, v(0)\big) = \varsigma\big(t, \Phi(\omega, v)\big).\end{aligned}$$

Then, M_0 turns out to be a minimal set for the scalar flow ς and, therefore, as stated in Theorem 3.1, it is an almost automorphic extension of Ω. It is immediate that then, so is M. Besides, observe that, whenever Ω is almost periodic, M is an almost automorphic minimal set. □

Finally, we want to make two remarks. First, if f is uniformly a.p. and (1) has a bounded solution on $\mathbb{R}^+$ (which gives rise to a relatively compact orbit in the skew-product semiflow), then for residually many equations (2) in the hull there is an a.a. solution. Second, under convexity assumptions on the function G in (7), the dynamical description of the a.p. and a.a. minimal sets can be deduced from [1] and [8].

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