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Incompressible nonlinearly elastic thin membranes

Karim Trabelsi

Laboratoire Jacques-Louis Lions, université Pierre et Marie Curie, boîte courrier 187, 75252 Paris cedex 05, France

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Abstract

Nonlinearly elastic thin membrane models are derived for hyperelastic incompressible materials using Γ-convergence arguments. We obtain an integral representation of the limit two-dimensional energy owing to a result of singular functionals relaxation due to Ben Belgacem [ESAIM Control Optim. Calc. Var. 5 (2000) 71–85 (electronic)]. ***To cite this article: K. Trabelsi, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Membranes minces non linéairement élastiques incompressibles. Des modèles de membranes minces non linéairement élastiques sont obtenus pour des matériaux hyperélastiques incompressibles via des arguments de Γ-convergence. Nous obtenons une représentation intégrale de l'énergie bidimensionnelle limite grâce à un résultat de relaxation de fonctionnelles singulières dû à Ben Belgacem [ESAIM Control Optim. Calc. Var. 5 (2000) 71–85 (électronique)]. ***Pour citer cet article : K. Trabelsi, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Version française abrégée

On considère un corps élastique mince de la forme $\Omega^\varepsilon = \omega \times]-\varepsilon, \varepsilon[$ dont la surface moyenne ω est un ouvert borné de $\mathbb{R}^2$ à frontière lipschitzienne et ε, son épaisseur, est le petit paramètre. On suppose que le corps est constitué d'un matériau hyperélastique incompressible dont la densité d'énergie est donnée par (1). Dans cette configuration la plaque subit une déformation $\varphi^\varepsilon : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$ qui devrait être une solution du problème (P^ε) où $J_\varepsilon(\varphi^\varepsilon) = I_\varepsilon(\varphi^\varepsilon) - \ell_\varepsilon(\varphi^\varepsilon)$, I_ε et ℓ_ε étant définies dans (2). Par ailleurs, on suppose que la fonction W obéit à des hypothèses de croissance et de coercivité données par (H). Par conséquence, l'espace des déformations admissibles est donné par (3) où la contrainte $\det \nabla\psi^\varepsilon = 1$ exprime l'incompressibilité du matériau. On remarque qu'aucune

E-mail address: trabelsi@ann.jussieu.fr (K. Trabelsi).

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hypothèse de convexité n'est faite sur W de sorte que la classe des énergies est assez grande pour inclure le matériau de Saint Venant–Kirchhoff, ainsi que les matériaux de type Ogden ; voir par exemple Ciarlet [6].

Le but de cette Note est d'étudier le comportement des suites minimisantes pour la suite d'énergies $(J_\varepsilon)_{\varepsilon>0}$ sur les espaces $(\mathcal{V}^\varepsilon)_{\varepsilon>0}$. Pour ce faire, il est utile de se ramener à un domaine fixe. A cet effet, on opère le changement d'échelle π_ε défini par $(\pi_\varepsilon f)(x^1,x^2,x^3) = f(x^1,x^2,\varepsilon x^3)$ et on note $I(\varepsilon)(\psi) = \varepsilon^{-1} I^\varepsilon(\pi_\varepsilon \psi)$, $\ell(\varepsilon)(\psi) = \varepsilon^{-1}\ell^\varepsilon(\pi_\varepsilon\psi)$ et $\bar{J}(\varepsilon) = I(\varepsilon) - \ell(\varepsilon)$. De même, on définit $\Omega = \Omega^1$, $\Gamma_\pm = \Gamma^1_\pm$,$\Gamma = \Gamma^1$, $U = U^1$ et $x = (x_H, \xi) \in \omega \times [-1,1]$. De la sorte, l'espace des déformations admissibles est donné par (4). Enfin, on suppose que $b(\varepsilon) = b$ et $g(\varepsilon) = \varepsilon g$ pour simplifier. Mise à l'échelle ainsi, une suite minimisante $(\psi(\varepsilon))_{\varepsilon>0} \subset \mathcal{V}(\varepsilon)$ de $\bar{J}(\varepsilon)$ vérifie (5) où h est une fonction positive telle que $h(\varepsilon) \to 0$ lorsque $\varepsilon \to 0$. Le comportement asymptotique d'une telle suite est décrit par la Γ-limite de la suite de fonctionnelles $(\bar{J}(\varepsilon))_{\varepsilon>0}$ par rapport à la topologie faible de $W^{1,p}(\Omega;\mathbb{R}^3)$; voir De Giorgi [8] et Dal Maso [7]. Cependant, pour éviter de travailler avec la topologie faible de $W^{1,p}(\Omega;\mathbb{R}^3)$ qui n'est pas métrisable sur les ensembles non-bornés, il est classique d'étendre les énergies à $L^p(\Omega;\mathbb{R}^3)$ comme dans (6).

Le résultat principal annoncé dans cette Note est la dérivation d'un modèle non linéaire membranaire de corps minces incompressibles par l'identification de la Γ-limite de la suite de fonctionnelles $(J(\varepsilon))_{\varepsilon>0}$. Introduisons d'abord l'espace $\mathcal{M}(\omega) = \{\bar{\varphi} \in W^{1,p}(\Omega;\mathbb{R}^3)\colon\ \bar{\varphi}_{,\xi} = 0 \text{ et } \bar{\varphi}_{|U}(x_H,\xi) = (x_H,0)\}$ des déformations membranaires. Alors on établit le Théorème 3.1. On remarque que l'énergie limite $J(0)$ ne dépend plus de la troisième variable ξ. De plus, la densité d'énergie $\mathbf{QR}W_0$ ne dépend que de la première forme fondamentale de la déformation. En ce sens, cette énergie décrit un corps élastique membranaire. La démonstration de ce théorème repose sur un résultat de Fonseca [9]. La motivation principale a été l'article [5] de Ben Belgacem concernant la relaxation de fonctionnelles singulières dont l'enveloppe rang-1-convexe est partout finie, ce qui est le cas pour W_0. Rappelons aussi que Ben Belgacem a aussi mené une analyse asymptotique semblable à la nôtre dans [3] pour un matériau compressible dont l'énergie vérifie $W(F) \to \infty$ si $\det F \to 0$; voir aussi [4]. La démonstration se fait par double inégalité ; c'est la démarche classique voir Le Dret et Raoult [11] et Acerbi et al. [1]. La borne inférieure est une conséquence aisée des propriétés des enveloppes rang-1-convexe et quasiconvexe. L'obtention de la borne supérieure est la difficulté principale. On commence par obtenir une borne supérieure valable pour des immersions, ensuite on passe par les applications affines par morceaux et localement injectives en affinant à chaque fois un peu plus la majoration avant d'aboutir à l'inégalite voulue. On fait aussi appel à un résultat de Kohn et Strang [10] et au théorème de recouvrement de Vitali.

Notons que l'énergie limite est séquentiellement semicontinue inférieurement faible dans $W^{1,p}(\omega;\mathbb{R}^3)$ par le théorème de relaxation dans [5] et aussi comme Γ-limite de fonctionnelles coercives. Une question naturelle s'impose : l'énergie bidimensionnelle $(\mathbf{QR}W)_0$ ne conviendrait-elle pas comme approximation ? N'éviterait-elle pas la relaxation ? La réponse est non, $\mathbf{R}W$ n'est pas finie et [5] n'est donc pas applicable. De plus, rien ne garantit que la fonctionnelle ainsi définie soit séquentiellement semicontinue inférieure faible dans $W^{1,p}(\omega;\mathbb{R}^3)$. Enfin, nous insistons sur le fait qu'il est inutile d'imposer une condition de convexité sur W puisqu'elle n'est pas en général héritée par W_0 ; voir par exemple Trabelsi [13].

Dans la Proposition 3.2, on montre que les propriétés essentielles requises pour W comme l'indifférence matérielle et l'isotropie sont transmises à $\mathbf{QR}W_0$. On remarque que comme dans le cas d'une énergie finie (Le Dret et Raoult [11]) ou singulière (Ben Belgacem [3,4]), la membrane incompressible ne résiste pas à la compression d'après le (iii) de la Proposition 3.2 ; voir aussi Trabelsi [12]

Les démonstrations complètes des résultats annoncés ainsi que d'autres remarques sont données dans Trabelsi [14].

1. Introduction

Let $(e_i)_{i=1,2,3}$ be an orthonormal basis of $\mathbb{R}^3$. Partial differentiation of a function or a vector field ψ with respect to the vector e_i is denoted $\psi_{,i}$ and its gradient (matrix) is denoted $\nabla\psi$. In its reference configuration, the body occupies the domain $\Omega^\varepsilon = \omega \times]-\varepsilon, \varepsilon[$, where the interior of its mid-surface ω is an open bounded subset of the plane spanned by e_1 and e_2 with Lipschitz-continuous boundary, and ε is the small parameter. The boundary of Ω^ε is partitioned as $\partial\Omega^\varepsilon = \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon \cup \Gamma^\varepsilon$, where $\Gamma_\pm^\varepsilon = \omega \times \{\pm\varepsilon\}$ and $\Gamma^\varepsilon = \partial\omega \times [-\varepsilon, \varepsilon]$. The body is submitted to dead body forces $b^\varepsilon \in L^q(\Omega^\varepsilon; \mathbb{R}^3)$ and to dead surface tractions $g_\pm^\varepsilon \in L^q(\Gamma_\pm^\varepsilon; \mathbb{R}^3)$, where $\frac{1}{p} + \frac{1}{q} = 1$ for instance, while the body is clamped on $U^\varepsilon = V \times [-\varepsilon, \varepsilon]$ where V is a neighbourhood of $\partial\omega$. The body is assumed to be made of an incompressible hyperelastic material whose stored energy function is given for all $F \in \mathbb{R}^{3\times3}$ by

$$W(F) = \begin{cases} \overline{W}(F) & \text{if } \det F = 1, \\ +\infty & \text{otherwise,} \end{cases} \tag{1}$$

where $\overline{W}$ is the finite elastic stored energy function. Such a material, when subjected to the loads and boundary conditions described above, undergoes a deformation $\varphi^\varepsilon : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$ which should be a stationary point of the energy (in particular a minimum) defined by $J^\varepsilon(\varphi^\varepsilon) = I^\varepsilon(\varphi^\varepsilon) - \ell^\varepsilon(\varphi^\varepsilon)$, where the functional I^ε measures the internal energy and the linear form ℓ^ε is the work of the external forces. In other words, we have

$$I^\varepsilon(\varphi^\varepsilon) = \int_{\Omega^\varepsilon} W(\nabla\varphi^\varepsilon)\,\mathrm{d}x \quad \text{and} \quad \ell^\varepsilon(\varphi^\varepsilon) = \int_{\Omega^\varepsilon} b^\varepsilon \cdot \varphi^\varepsilon\,\mathrm{d}x + \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} g^\varepsilon \cdot \varphi^\varepsilon\,\mathrm{d}x_H. \tag{2}$$

We make the following hypotheses on the function $\overline{W}$,

(i) (regularity) $\overline{W} : \mathbb{R}^{3\times3} \to \mathbb{R}$ is continuous,
(ii) (growth) There exist $C > 0$ and $p > 1$ such that $\overline{W}(F) \leqslant C(1 + |F|^p)$, for all $F \in \mathbb{R}^{3\times3}$, (H)
(iii) (coercivity) There exist $\alpha > 0$, $\beta \in \mathbb{R}$ and $p > 1$ such that $\overline{W}(F) \geqslant \alpha|F|^p + \beta$, for all $F \in \mathbb{R}^{3\times3}$.

Consequently, the manifold of admissible deformations is

$$\mathcal{V}^\varepsilon = \left\{\psi^\varepsilon \in W^{1,p}(\Omega^\varepsilon; \mathbb{R}^3):\ \det\nabla\psi^\varepsilon = 1 \text{ a.e. and } \psi^\varepsilon{}_{|U^\varepsilon} = \mathrm{id}\right\}, \tag{3}$$

where the constraint $\det\nabla\psi^\varepsilon = 1$ expresses the incompressibility of the material. Note that we make no convexity assumption on the function $\overline{W}$ so the class of functionals considered is large enough to include non-convex as well as polyconvex stored energy functions; see Ciarlet [6]. Now, the purpose of this Note is to examine the behaviour of the minimizing sequences of energies $(J^\varepsilon)_{\varepsilon>0}$ over the sets $(\mathcal{V}^\varepsilon)_{\varepsilon>0}$.

2. Asymptotic analysis

In order to carry out an asymptotic analysis, we first make a scaling that sets the problem onto a fixed domain. To this end, we define the operator π_ε by $(\pi_\varepsilon f)(x^1, x^2, \xi) = f(x^1, x^2, \varepsilon\xi)$ for every function f and we set $I(\varepsilon)(\psi) = \varepsilon^{-1} I^\varepsilon(\pi_\varepsilon\psi)$, $\ell(\varepsilon)(\psi) = \varepsilon^{-1}\ell^\varepsilon(\pi_\varepsilon\psi)$ and $\bar{J}(\varepsilon) = I(\varepsilon) - \ell(\varepsilon)$. We also define $\Omega = \Omega^1$, $\Gamma_\pm = \Gamma_\pm^1$, $\Gamma = \Gamma^1$, $U = U^1$ and $x = (x_H, \xi) \in \omega \times [-1, 1]$. Accordingly, we define the set

$$\mathcal{V}(\varepsilon) = \left\{\psi(\varepsilon) \in W^{1,p}(\Omega; \mathbb{R}^3):\ \det\nabla\psi(\varepsilon) = \varepsilon \text{ a.e. and } \psi(\varepsilon)_{|U} = \mathrm{id}\right\}. \tag{4}$$

Lastly, we assume that $b(\varepsilon) = b$ and $g(\varepsilon) = \varepsilon g$ for simplicity. Scaled in this fashion, a minimizing sequence $(\psi(\varepsilon))_{\varepsilon>0} \subset \mathcal{V}(\varepsilon)$ of $J(\varepsilon)$ verifies

$$\bar{J}(\varepsilon)(\psi(\varepsilon)) \leqslant \inf_{\phi \in \mathrm{id} + W_0^{1,p}(\Omega; \mathbb{R}^3)} \bar{J}(\varepsilon)(\phi) + h(\varepsilon) \quad \forall\varepsilon > 0, \tag{5}$$

where h is positive function such that $h(\varepsilon) \to 0$ when $\varepsilon \to 0$. Now, the characterization of the asymptotic behaviour of such a sequence is described by the Γ-limit of the sequence of functionals $(\bar{J}(\varepsilon))_{\varepsilon>0}$ with respect to the weak topology of $W^{1,p}(\Omega; \mathbb{R}^3)$; see De Giorgi [8] and Dal Maso [7]. However, to avoid the use of the weak topology of

$W^{1,p}(\Omega;\mathbb{R}^3)$ which is not metrizable on unbounded sets, a classical trick is to extend the energies to $L^p(\Omega;\mathbb{R}^3)$ by setting

$$J(\varepsilon)\big(\psi(\varepsilon)\big)=\begin{cases}\bar{J}(\varepsilon)\big(\psi(\varepsilon)\big) & \text{if } \psi(\varepsilon)\in\mathcal{V}(\varepsilon),\\ +\infty & \text{if } \psi\in L^p(\Omega;\mathbb{R}^3) \text{ and } \psi\notin\mathcal{V}(\varepsilon).\end{cases}\tag{6}$$

3. Main result

The main result of this Note is the derivation of a nonlinear membrane body model made of incompressible material through the identification of the Γ-limit of the sequence of functionals $(J(\varepsilon))_{\varepsilon>0}$. Let us first introduce the set of admissible membrane deformations $\mathcal{M}(\omega)=\{\varphi\in W^{1,p}(\omega;\mathbb{R}^3)\colon\ \varphi_{|V}(x_H)=(x_H,0)\}\approx\{\bar{\varphi}\in W^{1,p}(\Omega;\mathbb{R}^3)\colon\ \bar{\varphi}_{,\xi}=0 \text{ and } \bar{\varphi}_{|U}(x_H,\xi)=(x_H,0)\}$. As expected, admissible membrane deformations are no longer dependent on the third variable. We now state our main result.

Theorem 3.1. *The functionals $J(\varepsilon)$ are Γ-convergent towards $J(0)$ with respect to the weak topology of $W^{1,p}(\Omega;\mathbb{R}^3)$ where the functional $J(0)$ is defined by*

$$J(0)(\varphi)=\begin{cases}2\displaystyle\int_\omega \mathbf{QR}W_0(\nabla\varphi)\,\mathrm{d}x_H-\ell(0) & \text{if } \varphi\in\mathcal{M}(\omega),\\ +\infty & \text{otherwise},\end{cases}\tag{7}$$

and

$$W_0(F)=\inf_{z\in\mathbb{R}^3}W\big((F|z)\big)\quad \text{and}\quad \ell(0)(\varphi)=\int_\omega\left\{\int_{-1}^{1}b(x_H,\xi)\,\mathrm{d}\xi+g(x_H,-1)+g(x_H,1)\right\}\cdot\varphi\,\mathrm{d}x_H.\tag{8}$$

We sketch the proof of the above theorem in the next section. Before that, we state some basic properties that the two-dimensional energy $\mathbf{QR}W_0$ inherits from the three-dimensional energy W.

Proposition 3.2.

(i) *If the function $\overline{W}$ satisfies the principle of material frame-indifference, then the function $\mathbf{QR}W_0$ is frame-indifferent, i.e.,*

$$\mathbf{QR}W_0(F)=\mathbf{QR}W_0(RF)\quad \forall R\in \mathrm{SO}(3),\ F\in\mathbb{R}^{3\times 2}.\tag{9}$$

(ii) *If the function $\overline{W}$ is isotropic, then the function $\mathbf{QR}W_0$ is isotropic, i.e.,*

$$\mathbf{QR}W_0(F)=\mathbf{QR}W_0(F\overline{R})\quad \forall \overline{R}\in \mathrm{SO}(2),\ F\in\mathbb{R}^{3\times 2}.\tag{10}$$

(iii) *If the function $\overline{W}$ is frame-indifferent and isotropic, then there exists a symmetric function $\phi:\mathbb{R}^2_+\to\mathbb{R}$ such that*

$$\mathbf{QR}W_0(F)=\phi\big(s_1(F),s_2(F)\big)\tag{11}$$

where $s_1(F)$ and $s_2(F)$ are the singular values of matrix $F\in\mathbb{R}^{3\times 2}$. If in addition $\overline{W}$ is positive and verifies $\overline{W}(\mathrm{id})=0$, then $\phi(x,y)=0$ for all $(x,y)\in[0,1]^2$.

4. Sketch of the proof

The proof of Theorem 3.1 is carried out by establishing a “double inequality”; this is the classical procedure followed by Le Dret and Raoult [11] and Acerbi et al. [1]. Indeed, the lower bound of the Γ-limit is a straightforward consequence of the properties of the quasiconvex and rank-one-convex envelopes. The obtention of the upper bound is a more arduous task. In effect, we start by showing that if $\psi \in C^1(\overline{\omega};\mathbb{R}^3)$ is an immersion such that $\psi_{|V} = \mathrm{id}$ and $\theta \in C(\overline{\omega};\mathbb{R}^3)$ is a function such that $\theta_{|V} = e_3$ and $\det(\psi_{,1}|\psi_{,2}|\theta) = 1$, then the following inequality holds

$$\bar{J}(0)(\psi) \leqslant 2\int_\omega W\big((\psi_{,1}|\psi_{,2}|\theta)\big)\,\mathrm{d}x_H - \ell(0)(\psi). \tag{12}$$

The above objective is achieved by constructing a sequence of deformations in $\mathcal{V}(\varepsilon)$ that converges to ψ in $W^{1,p}(\Omega;\mathbb{R}^3)$. Then by an approximation lemma due to Ben Belgacem [3], which roughly states that locally injective piecewise affine functions can be approximated by a sequence of immersions whose singular values remain larger than a strictly positive real thereby enabling the use of Lebesgue’s dominated convergence theorem, we refine the upper bound and prove that

$$J(0)(\psi) \leqslant 2\int_\omega W_0(\nabla\psi)\,\mathrm{d}x_H - \ell(0)(\psi), \tag{13}$$

for all locally injective function $\psi \in \mathrm{Aff}(\omega;\mathbb{R}^3)$ such that $\psi_{|V} = \mathrm{id}$, where $\mathrm{Aff}(\omega;\mathbb{R}^3)$ is the class of piecewise affine functions defined on ω and into $\mathbb{R}^3$. To prove inequalities (12) and (13), it is convenient to express function θ in the local coordinate system defined by the tangent plane at each point of the body: $\theta = \mu\psi_{,1} + \nu\psi_{,2} + \lambda\ (\psi_{,1}\wedge\psi_{,2})$ since $\det\nabla\psi^t\nabla\psi > 0$. Next, we move on to the relaxation of the last bound. The first step is to show, by induction, that for all locally injective function $\psi \in \mathrm{Aff}(\omega;\mathbb{R}^3)$ such that $\psi_{|V} = \mathrm{id}$, we have

$$J(0)(\psi) \leqslant 2\int_\omega \mathrm{R}_k W_0(\nabla\psi)\,\mathrm{d}x_H - \ell(0)(\psi), \quad k \in \mathbb{N}, \tag{14}$$

where $(\mathrm{R}_k W_0)_{k\in\mathbb{N}}$ is the Kohn and Strang [10] sequence that converges to $\mathbf{R}W_0$. Inequality (14) is obtained by using a result due to Fonseca [9] which states that if $D \subset \mathbb{R}^2$ is open and bounded, the function $\mathrm{Z}_D W_0$ defined by

$$\mathrm{Z}_D W_0(F) = \frac{1}{|D|}\inf\bigg\{\int_D W_0(F+\nabla\varphi)\colon\ \varphi \in \mathrm{Aff}(\omega;\mathbb{R}^3)\bigg\}, \quad F \in \mathbb{R}^{3\times 2}, \tag{15}$$

is rank-one-convex on its effective domain, i.e., the subset of $\mathbb{R}^{3\times 2}$ where it is finite; see also Ben Belgacem [5]. Another core ingredient is Vitali’s covering theorem. The next move is to take the limit as $k \to \infty$. After showing that immersions can be approximated by locally injective piecewise affine functions, we thus infer that for all immersions $\psi \in C^1(\overline{\omega};\mathbb{R}^3)$ such that $\psi_{|V} = \mathrm{id}$, we have

$$J(0)(\psi) \leqslant 2\int_\omega \mathbf{R}W_0(\nabla\psi)\,\mathrm{d}x_H - \ell(0)(\psi). \tag{16}$$

Lastly we invoke the density of immersions in $W^{1,p}(\omega;\mathbb{R}^3)$ (cf. Ben Belgacem [3]) to derive the above bound for deformations in $W^{1,p}(\omega;\mathbb{R}^3)$. After proving that function $\mathbf{R}W_0$ is everywhere finite, we use Acerbi and Fusco [2] to get the announced result.

5. Commentaries

The asymptotic analysis carried out here essentially relies on Ben Belgacem's result [5]. One of the limitations of this result is that it requires that the rank-one-convex envelope of the functional be everywhere finite, which does not seem to be an impediment for most cases of stored energy functions arising in elasticity, in particular for W_0. However, his result does not apply to the function W since $\mathbf{R}W$ can be infinite as there are no rank-one-connected matrices with determinant equal to 1. Consequently, the stored energy function $(\mathbf{QR}W)_0$ is not a valid candidate for a membrane model. Besides, it is of no use to impose any convexity condition on W since it is not, in general, inherited by W_0; for instance see Trabelsi [13].

We also remark that the limit functional $\bar{J}(0)$ admits minimizers in $\mathcal{M}(\omega)$. This claim is vindicated either by the functional being a Γ-limit of coercive functionals (cf. [7]) or by the $W^{1,p}(\omega;\mathbb{R}^3)$ weak lower semicontinuity of the limit functional and its coercivity and growth conditions according to [5].

To conclude, we mention that Ben Belgacem [3,4] has led a similar asymptotic analysis for compressible materials such that $W(F) \to \infty$ if $\det F \to 0$. These results are comparable to ours. Note that as in the finite energy case (Le Dret and Raoult [11]) or the singular case (Ben Belgacem [3,4]), the incompressible membrane does not resist to compression in view of (iii) in Proposition 3.2; see also Trabelsi [12].

For detailed proofs of the results stated in this Note and further remarks we refer to Trabelsi [14].

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