Available online at www.sciencedirect.com
SCIENCE DIRECT
C. R. Acad. Sci. Paris, Ser. I 340 (2005) 113–118
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Partial Differential Equations/Mathematical Problems in Mechanics

Existence of weak solutions for an interaction problem between an elastic structure and a compressible viscous fluid

Muriel Boulakia

Laboratoire de mathématiques appliquées, université de Versailles-St-Quentin, 45, avenue des Etats Unis, 78035 Versailles cedex, France

Received 18 July 2004; accepted after revision 29 October 2004

Available online 24 December 2004

Presented by Pierre-Louis Lions

Abstract

We prove an existence result of weak solutions for an interaction problem between an elastic structure and a compressible fluid in three space dimensions. Solutions are defined as long as there is no collision and as long as conditions of non-interpenetration and of preservation of orientation are satisfied by the displacement field of the structure. ***To cite this article: M. Boulakia, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Existence d'une solution faible pour un problème d'interaction fluide visqueux compressible-solide élastique. Nous présentons ici un résultat d'existence de solutions faibles pour un problème d'interaction entre une structure élastique et un fluide compressible en dimension trois. Les solutions sont définies tant qu'il n'y pas de chocs et tant que le déplacement de la structure vérifie des conditions de non-interpénétration et de préservation de l'orientation. ***Pour citer cet article : M. Boulakia, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Version française abrégée

On considère une structure élastique immergée dans un fluide visqueux barotropique compressible en dimension trois. L'ensemble évolue dans une cavité fixe bornée Ω.

L'évolution de la vitesse fluide est donnée par les équations de Navier–Stokes compressibles et on prend comme loi d'état la loi barotropique avec une constante adiabatique $\gamma > \frac{3}{2}$. Sur la partie solide, on choisit de garder le point de vue eulérien qui intervient naturellement sur la partie fluide. Le flot solide est donc une inconnue du problème

E-mail address: boulakia@math.uvsq.fr (M. Boulakia).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2004.11.003

exprimée en fonction de la vitesse eulérienne solide. La loi de comportement du solide est la loi non linéaire de Saint Venant–Kirchhoff. Dans l'équation de la quantité de mouvement du solide, on ajoute un terme régularisant qui permet de contrôler le flot en norme suffisamment forte. Ce problème est complété par des conditions de couplage et de bord. Enfin, la densité globale satisfait l'équation de continuité.

Dans ce travail, on montre un résultat d'existence de solution faible tant qu'il n'y pas de chocs et tant que le déplacement de la structure vérifie des conditions de non-interpénétration et de préservation de l'orientation. On fait référence à [1] pour des explications complémentaires. Notre résultat est montré en reprenant la méthode introduite par [5]. Cet article traite du cas d'un fluide compressible seul. La méthode repose sur deux régularisations successives : on considère une loi barotropique avec une constante adiabatique plus élevée (voir (15)) et on ajoute un terme visqueux dans l'équation de continuité sur la partie fluide de façon à remplacer cette équation par une équation parabolique de type chaleur (voir (12)). La première étape consiste donc à montrer que ce problème régularisé admet une solution. A ce niveau, on retrouve les difficultés habituelles des problèmes fluide-structure : les domaines solide et fluide sont inconnus. Pour surmonter cela, on utilise un argument de point fixe sur un problème linéarisé en dimension finie. Le choix d'adopter un point de vue eulérien aussi bien sur la partie solide que sur la partie fluide simplifie tout de même l'écriture du problème.

Pour les passages à la limite successifs, par rapport à l'article [5], des difficultés supplémentaires apparaissent dues au fait que l'interface est mobile. En particulier, pour obtenir des estimations supplémentaires sur la pression, on utilise la méthode développée dans [8].

1. Introduction and equations of motion

We study the three dimensional motion of an elastic structure immersed in a viscous compressible fluid. The fluid and the structure are contained in a fixed bounded set Ω. We denote by $\Omega_S(t)$ the domain occupied by the structure and $\Omega_F(t) = \Omega \setminus \overline{\Omega_S(t)}$ the domain occupied by the fluid at time t. In the fluid, the Eulerian velocity u_F and the density ϱ_F satisfy the Navier–Stokes equations:

$$\partial_t(\varrho_F u_F) + \operatorname{div}(\varrho_F u_F \otimes u_F) + \nabla p - \operatorname{div} \mathbb{T} = 0 \quad \text{in } \Omega_F(t) \tag{1}$$

where the stress tensor $\mathbb{T}$ is defined by: $\mathbb{T} = \mu_F \nabla u_F + (\lambda_F + \mu_F) \operatorname{div} u_F \operatorname{Id}$.

The viscosity coefficients λ_F and μ_F are such that: $\mu_F > 0, \lambda_F + 2\mu_F > 0$ and p denotes the pressure. The relation between the pressure and the density is given by the constitutive law, $p = a\varrho_F^\gamma$, where a is a strictly positive constant and $\gamma > \frac{3}{2}$ is the adiabatic constant. On the structure, we choose to keep this Eulerian notation. Let u_S be the Eulerian velocity of the structure, ϱ_S the structure density and X_S the displacement field. For all y in $\Omega_S(0)$, for all t in $[0, T]$, $X_S(t, 0, y)$ is the position at time t of the particle located in y at initial time. The relation between u_S and X_S is: for all $y \in \Omega_S(0)$,

$$\begin{cases} \partial_t X_S(t, 0, y) = u_S\big(t, X_S(t, 0, y)\big), \\ X_S(0, 0, y) = y. \end{cases} \tag{2}$$

We consider the following equation of linear momentum:

$$\partial_t(\varrho_S u_S) + \operatorname{div}(\varrho_S u_S \otimes u_S) + \theta A_3 u_S - \operatorname{div} \sigma_S = 0 \quad \text{in } \Omega_S(t). \tag{3}$$

The term $\theta A_3 u_S$ is a regularizing term which is necessary for our study. A_3 is a differential operator independent of time such that:

$$\int_{\Omega_S(t)} \big(A_3 u(y), v(y)\big)\,\mathrm{d}y = (u, v)_{H^3(\Omega_S(t))}, \quad \forall u, v \in \mathcal{D}\big(\Omega_S(t)\big)^3$$

and θ is a fixed strictly positive real number. Thanks to this regularization, we control the norm of the displacement field X_S in $H^1(0,T;W^{1,\infty}(\Omega_S(0)))$. The Cauchy stress tensor σ_S is expressed with respect to the second Piola–Kirchhoff tensor $\hat{\hat{\sigma}}$:

$$\sigma_S(t,x)=\det\nabla X_S(0,t,x)\nabla X_S(0,t,x)^{-1}\hat{\hat{\sigma}}_S\big(t,X_S(0,t,x)\big)\nabla X_S(0,t,x)^{-t},\quad \forall x\in\Omega_S(t)$$

and the constitutive law is the Saint Venant–Kirchhoff law $\hat{\hat{\sigma}}_S[X_S]=2\mu_S E(X_S)+\lambda_S\,\mathrm{tr}(E(X_S))\,\mathrm{Id}$ where the Lamé constants of the elastic media λ_S and μ_S satisfy: $\mu_S>0, \lambda_S+2\mu_S>0$ and $E(X_S)$ is the Green–Saint Venant tensor defined by $E(X_S)=\frac{1}{2}({}^t\nabla X_S\nabla X_S-\mathrm{Id})$.

This system is completed by boundary conditions. As the fluid is viscous, the velocity is continuous at the interface:

$$\begin{cases} u_F=0 & \text{on } \partial\Omega,\\ u_F=u_S & \text{on } \partial\Omega_S(t). \end{cases} \tag{4}$$

The second equation is a coupling equation between the fluid and the structure. The coupling is also expressed by the continuity of the stress on the interface: $\forall t\in[0,T]$, $\forall v\in\mathcal{C}(\partial\Omega_S(t))$,

$$\int_{\partial\Omega_S(t)}(\mathbb{T}-p\,\mathrm{Id})n_x\cdot v=\int_{\partial\Omega_S(t)}\sigma_S n_x\cdot v-\theta\langle u_S,v\rangle_{3,\partial\Omega_S(t)} \tag{5}$$

where the operator $\langle\cdot,\cdot\rangle_{3,\partial\Omega_S(t)}$ represents the contributing terms on the boundary of the operator A_3: $\forall u,v\in\mathcal{D}(\overline{\Omega_S(t)})$, $\int_{\Omega_S(t)}A_3uv=(u,v)_{H^3(\Omega_S(t))}+\langle u,v\rangle_{3,\partial\Omega_S(t)}$. We denote by u the global Eulerian velocity and by ϱ the global density defined on Ω. The evolution of ϱ is given by the continuity equation:

$$\partial_t\varrho+\mathrm{div}(\varrho u)=0\quad \text{in } \Omega. \tag{6}$$

Next, we define the concept of *renormalized solutions* introduced in [3] (see also [7]) with slightly modified conditions on the admissible functions b:

Definition 1.1. The continuity Eq. (6) is satisfied in the sense of *renormalized solutions* if, for any $b\in\mathcal{C}^1(\mathbb{R})$ such that $b'(z)=0$ for z large enough, we have:

$$\partial_t b(\varrho)+\mathrm{div}\big(b(\varrho)u\big)+\big(b'(\varrho)\varrho-b(\varrho)\big)\,\mathrm{div}\,u=0\quad \text{in } \mathcal{D}'\big((0,T)\times\Omega\big). \tag{7}$$

Our system is completed by initial conditions:

$$u(t=0)=u^0\quad \text{in } \Omega,\qquad \varrho(t=0)=\varrho^0=\begin{cases}\varrho_S^0 & \text{in } \Omega_S(0),\\ \varrho_F^0 & \text{in } \Omega_F(0).\end{cases}$$

2. Variational formulation

Let $\mathcal{V}$ be the test function space:

$$\mathcal{V}=\big\{v\in\mathcal{C}^\infty\big((0,T)\times\Omega\big)^3 \mid v(T)=0,\ v(t,\cdot)\in H_0^1(\Omega)^3,\ \forall t\in[0,T]\big\}. \tag{8}$$

Definition 2.1. We will say that (X_S,ϱ,u) is a weak solution of the problem (1)–(6) if:

(i) $X_S\in H^1(0,T;H^3(\Omega_S(0)))^3$, $\varrho\in L^\infty(0,T,L^\gamma(\Omega))$, $\varrho\geqslant 0$, $u\in L^2(0,T,H_0^1(\Omega))^3$,
(ii) Eq. (2) is satisfied almost everywhere on $(0,T)\times\Omega_S(0)$,
(iii) the continuity equation (6) is satisfied in the sense of renormalized solutions,

(iv) the following weak formulation holds: for all $v \in \mathcal{V}$,

$$\int_0^T\int_\Omega \varrho u\cdot\partial_t v\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega \varrho(u\otimes u):\nabla v\,\mathrm{d}x\,\mathrm{d}tn^o-\int_0^T\int_{\Omega_S(t)}\sigma_S:\nabla v-\theta\int_0^T\big(u(t),v(t)\big)_{H^3(\Omega_S(t))}\,\mathrm{d}t$$
$$-\int_0^T\int_{\Omega_F(t)}\mathbb{T}:\nabla v\,\mathrm{d}x\,\mathrm{d}tn^o+a\int_0^T\int_{\Omega_F(t)}\varrho_F^\gamma\,\mathrm{div}\,v\,\mathrm{d}x\,\mathrm{d}t=-\int_\Omega \varrho^0u^0\cdot v(0,\cdot)\,\mathrm{d}y. \qquad (9)$$

3. Main result

Theorem 3.1. *Let $u^0\in H_0^1(\Omega)^3$, $\rho_S^0\in L^\infty(\Omega)$ and $\varrho_F^0\in L^\gamma(\Omega_F(0))$ satisfying*:

$$0<\underline{\varrho}_S\leqslant\varrho_S^0(x)\leqslant\bar{\varrho}_S,\quad \forall x\in\Omega_S(0)\quad\textit{and}\quad \varrho_F^0(x)\geqslant 0,\quad\forall x\in\Omega_F(0). \qquad (10)$$

We define: $d(t)=d(\partial\Omega_S(t),\partial\Omega)$ *and* $\gamma(t)=\inf_{y\in\Omega_S(0)}|\det\nabla X_S(t,0,y)|$ *and we suppose that* $d(0)>0$. *Then, there exists at least one weak solution of the problem* (1)–(6) *defined on* $(0,T)$ *where*:

$$T=\sup\big\{t>0\mid d(t)>0,\gamma(t)>0\textit{ and }X_S(t,0,\cdot)\textit{ one-to-one}\big\}$$

(T *is bounded away from* 0 *by an explicit constant depending on the data and* θ).

Moreover, denoting E_0 the initial energy, this solution satisfies the energy estimate:

$$\frac12\int_\Omega\varrho(t)\big|u(t)\big|^2\,\mathrm{d}x+\frac{a}{\gamma-1}\int_{\Omega_F(t)}\varrho_F^\gamma(t)+\mu_F\int_0^t\int_{\Omega_F(s)}\big|\nabla u_F\big|^2n^o+(\lambda_F+\mu_F)\int_0^t\int_{\Omega_F(s)}\big|\mathrm{div}\,u_F\big|^2$$
$$+\theta\int_0^t\|u_S\|^2_{H^3(\Omega_S(s))}n^o+\mu_S\int_{\Omega_S(0)}\big|E\big(X_S(t,0,y)\big)\big|^2\,\mathrm{d}y+\frac{\lambda_S}{2}\int_{\Omega_S(0)}\big|\mathrm{tr}\,E\big(X_S(t,0,y)\big)\big|^2\,\mathrm{d}y\leqslant E_0. \qquad (11)$$

To demonstrate this theorem, we follow the method introduced in [5] which proves the global existence of weak solutions to the compressible Navier–Stokes equations. We give here a sketch of the proof. For a complete proof, we refer to [1]. The method presented in [5] has already been adapted in the case of a rigid structure immersed in a compressible fluid in [4]. For other studies dealing with interaction between a compressible fluid and a structure we refer also to [6] for an elastic structure and [2] for a rigid structure.

4. A regularized problem

We first solve a regularized problem. On the fluid domain, we add an artificial viscosity term in the continuity equation:

$$\begin{cases}\partial_t\varrho_F+\mathrm{div}(\varrho_F u)=\epsilon\Delta\varrho_F & \text{in }\Omega_F(t),\\ \nabla\varrho_F\cdot n=0 & \text{on }\partial\Omega_F(t)\end{cases} \qquad (12)$$

where $\epsilon>0$ is small. On the structure domain, we keep the initial equation:

$$\partial_t\varrho_S+\mathrm{div}(\varrho_S u)=0\quad\text{in }\Omega_S(t). \qquad (13)$$

We require some regularity on the initial conditions in order to obtain regularity results on the problem (12): we consider initial data $\varrho_F^0 \in H^2(\Omega_F(0))$ and $\varrho_S^0 \in L^\infty(\Omega_S(0))$ such that $0 < \underline{\varrho} \leqslant \varrho^0(x) \leqslant \bar{\varrho}$, $\forall x \in \Omega$. Moreover, we consider the following system for modeling the fluid motion:

$$\partial_t(\varrho_F u_F) + \operatorname{div}(\varrho_F u_F \otimes u_F) + \epsilon \nabla u_F \nabla \varrho_F + \nabla p - \operatorname{div} \mathbb{T} = 0 \quad \text{in } \Omega_F(t) \tag{14}$$

where the pressure p is now defined by:

$$p = a\varrho_F^\gamma + \delta \varrho_F^\beta \tag{15}$$

where $\delta > 0$ is small and $\beta > 0$ is sufficiently large. The first step consists in proving the existence of a weak solution to the variational formulation associated to (2), (3), (12), (13) and (14) completed by the relations (4) and (5).

As is often the case in fluid–structure interaction problems, we are not able to solve this problem directly; we use a linearization procedure. We first solve a linearized finite dimensional problem and then, thanks to a fixed point argument, we obtain an approximate solution in finite dimension (X_S^N, ϱ^N, u^N) which satisfies an energy estimate. We do not detail here the construction of the linearized problem. To pass to the limit when N goes to infinity, we need compactness results. First of all, thanks to the regularization term for the structure velocity, we obtain that $X^N \to X$ in $\mathcal{C}(0,T;\mathcal{C}^1(\Omega_S(0)))$. Using an adaptation of Aubin's lemma for moving domains, we show that (ϱ^N) strongly converges in $L^\beta((0,T)\times\Omega)$. As $(\sqrt{\varrho^N} u^N)$ is bounded in $L^\infty(0,T;L^2(\Omega))$ and (ϱ^N) is bounded in $L^\infty(0,T;L^\gamma(\Omega))$, $(\varrho^N u^N)$ is bounded in $L^\infty(0,T;L^{2\gamma/(\gamma+1)}(\Omega))$ and converges to ϱu. This allows us to pass to the limit in the continuity equation. We also show, thanks to a regularity result on the parabolic equations, that the limit ϱ satisfies the continuity equation almost everywhere. From the variational problem satisfied by u^N, we strengthen the convergence in time of $(\varrho^N u^N)$:

$$\varrho^N u^N \to \varrho u \quad \text{in } \mathcal{C}\big(0,T;L_w^{2\gamma/(\gamma+1)}(\Omega)\big).$$

As $\gamma > \frac{3}{2}$, the embedding $L^{2\gamma/(\gamma+1)}(\Omega) \subset H^{-1}(\Omega)$ is compact. Therefore, we conclude that $\varrho^N u^N \otimes u^N$ weakly converges to $\varrho u \otimes u$ in $\mathcal{D}'((0,T)\times\Omega)$. All these results allow us to pass to the limit in N in the variational formulation and we obtain, in this way, a finite energy solution of our regularized problem.

5. Passing to the limit in the viscosity term

We want now to pass to the limit in ϵ. Similarly as above, we have the strong convergence of the sequence (X_ϵ) in $\mathcal{C}(0,T;\mathcal{C}^1(\Omega_S(0)))$. As (ϱ_ϵ) weakly converges in $L^\infty(0,T;L^\gamma(\Omega))$, thanks to the continuity equation, we can strengthen this convergence: (ϱ_ϵ) converges in $\mathcal{C}(0,T;L_w^\gamma(\Omega))$ to ϱ. Making use of the compactness of $L^\gamma(\Omega)$ in $H^{-1}(\Omega)$, we deduce that (ϱ_ϵ) strongly converges in $\mathcal{C}(0,T;H^{-1}(\Omega))$. As (u_ϵ) is bounded in $L^2(0,T;H_0^1(\Omega))$, we conclude that the sequence $(\varrho_\epsilon u_\epsilon)$ converges weakly in $L^\infty(0,T;L^{2\gamma/(\gamma+1)}(\Omega))$ to ϱu. Then, we identify the limit of $\varrho_\epsilon u_\epsilon \otimes u_\epsilon$ similarly as in the previous section.

To pass to the limit in the variational formulation, it remains to identify the term of pressure. To do this, we need complementary estimates on the fluid density: we show that the sequence $(\varrho_{F,\epsilon})$ is bounded in $L^{\beta+1}((0,T)\times\Omega)$. At this point, some difficulties to obtain estimates up to boundary come out from the fact that we deal with a mobile interface. Estimates on the density are obtained following the method of [8]. At this step, we need regularity results on the Stokes problem defined on the open set $\Omega_{F,\epsilon}(t)$.

Next, we introduce the effective viscous flux $a\varrho_{F,\epsilon}^\gamma + \delta\varrho_{F,\epsilon}^\beta - (\lambda_F + 2\mu_F)\operatorname{div} u_\epsilon$ and we show that: for any $\varphi \in \mathcal{D}(0,T_0;\mathcal{D}(\Omega_F(t)))$,

$$\lim_{\epsilon\to 0} \int_0^{T_0}\!\!\int_\Omega \varphi^2\big(a\varrho_{F,\epsilon}^\gamma + \delta\varrho_{F,\epsilon}^\beta - (\lambda_F + 2\mu_F)\operatorname{div} u_\epsilon\big)\varrho_{F,\epsilon} = \int_0^{T_0}\!\!\int_\Omega \varphi^2\big(p - (\lambda_F + 2\mu_F)\operatorname{div} u\big)\varrho_F$$

where p is the limit of $(a\varrho_{F,\epsilon}^{\gamma} + \delta\varrho_{F,\epsilon}^{\beta})$. Here, again, we follow the proof of [5]. Thanks to a regularization procedure [3], we show that in fact ϱ is a renormalized solution of the continuity equation. Taking $b = z\log z$ in (7) and in the renormalized equation satisfied by ϱ_ϵ, we prove that:

$$\lim_{\epsilon\to 0}\int_0^{T_0}\int_{\Omega_{F,\epsilon}(t)} \operatorname{div} u_{\mathrm{e}}\varrho_\epsilon \leqslant \int_0^{T_0}\int_{\Omega_F(t)} \operatorname{div} u\varrho.$$

These two results, added to a monotony argument, allow us to conclude that $p = a\varrho_F^{\gamma} + \delta\varrho_F^{\beta}$.

6. Passing to the limit in the artificial pressure term

It remains to pass to the limit in δ. At this stage, we can relax the regularity of the initial data prescribed for the fluid density. We can follow the arguments developed previously. To identify the pressure, thanks to the method introduced in [8], we show that (ϱ_δ) stays bounded in $L^{\gamma+\alpha}((0,T_0)\times\Omega)$ with $\alpha > 0$. The end of the proof is very close to the demonstration given in [5]. To pursue, we introduce cut-off operators T_k. Denoting $\overline{T_k(\varrho)}$ the limit of the sequence $(T_k(\varrho_\delta))$, we prove $\limsup_{\delta\to 0}\|T_k(\varrho_\delta) - T_k(\varrho)\|_{L^{\gamma+1}((0,T_0)\times\Omega)}$ stay bounded and consequently in $L^2((0,T_0)\times\Omega)$. This allows us to prove that ϱ is a renormalized solution of the continuity equation and to identify the pressure.

To conclude, we prove that we can extend our solution beyond T_0. Thanks to the energy estimate satisfied by our solution, we iterate the same proof with the new configuration references $\Omega_S(T_0)$ and $\Omega_F(T_0)$ and we show that our solution is defined on a time interval of fixed length independent of T_0. This allows to conclude the proof of the theorem.

Acknowledgements

The author wants to thank Pr. Jean-Pierre Puel who introduced her to this subject and gave her precious advices and Pr. Eduard Feireisl for fruitful discussions.

References

[1] M. Boulakia, Modélisation et analyse mathématique de problèmes d'interaction fluide–structure, Thesis, université de Versailles, 2004.
[2] B. Desjardins, M.J. Esteban, On weak solutions for fluid-rigid structure interaction: compressible and incompressible models, Commun. Partial Differential Equations 25 (7–8) (2000) 1399–1413.
[3] R.J. Di Perna, P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math. 98 (1989) 511–547.
[4] E. Feireisl, On the motion of rigid bodies in a viscous compressible fluid, Arch. Rational Mech. Anal. 167 (4) (2003) 281–308.
[5] E. Feireisl, A. Novotnỳ, H. Petzeltovà, On the existence of globally defined weak solutions to the Navier–Stokes equations, J. Math. Fluid Mech. 3 (4) (2001) 358–392.
[6] F. Flori, P. Orenga, Fluid–structure interaction: analysis of a 3-D compressible model, Ann. Inst. H. Poincaré Anal. Non Linéaire 17 (6) (2000) 753–777.
[7] P.L. Lions, Mathematical Topics in Fluid Mechanics, Oxford Science Publications, 1996.
[8] P.L. Lions, Bornes sur la densité pour les équations de Navier–Stokes compressibles isentropiques avec conditions aux limites de Dirichlet, C. R. Acad. Sci. Paris, Ser. I 328 (8) (1999) 659–662.