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Mathematical Analysis/Harmonic Analysis

Pointwise regularity criteria

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Abstract

A wavelet characterization of the pointwise regularity condition $T_u^p(x_0)$ of Calderón and Zygmund is obtained. The extremal case (a pointwise BMO condition) yields the sharpest wavelet condition which is implied by pointwise Hölder regularity; in particular, this criterium is sharper than the usual two-microlocal condition. ***To cite this article: S. Jaffard, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Critères de régularité ponctuelle. On obtient une caractérisation par ondelettes de la condition de régularité ponctuelle $T_u^p(x_0)$ de Calderón et Zygmund. Le cas extrème (une condition de type BMO local) fournit la condition la plus précise sur les modules des coefficients d'ondelette impliquée par la régularité Hölderienne ponctuelle ; en particulier elle est plus fine que le critère deux-microlocal usuel. ***Pour citer cet article : S. Jaffard, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Il existe plusieurs définitions possibles de la régularité ponctuelle d'une fonction $f:\mathbb{R}^d \to \mathbb{R}$; la plus couramment utilisée est la *régularité Hölderienne*.

Définition 0.1. Soient $f \in L^\infty_{\mathrm{loc}}$, $x_0 \in \mathbb{R}^d$ et $\alpha \geqslant 0$; alors $f \in C^\alpha(x_0)$ s'il existe $R > 0$, $C > 0$, et un polynôme P de degré inférieur à α tels que

$$\text{si } |x - x_0| \leqslant R \text{ alors } \big|f(x) - P(x - x_0)\big| \leqslant C|x - x_0|^\alpha. \tag{1}$$

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La condition L^∞_{loc} est nécessaire ; en effet (1) implique que f est bornée au voisinage de x_0. Donc cette définition n'est pas utilisable si le cadre naturel pour f est L^p_{loc} pour un $p < \infty$. Un autre inconvénient a été découvert par Calderón et Zygmund en 1961, cf. [2] : la condition (1) n'est pas conservée sous l'action des opérateurs pseudodifférentiels classiques d'ordre 0. De plus, les ondelettes ne sont pas des bases inconditionelles de l'espace $C^\alpha(x_0)$, cf. [3]. Aussi Calderón et Zygmund ont introduit les conditions de régularité ponctuelle $T^p_u(x_0)$ qui ne souffrent pas de ces inconvénients. Soit $p \in (1,\infty)$, $f \in L^p_{\text{loc}}$ et $u \geqslant -d/p$; on dit que $f \in T^p_u(x_0)$ s'il existe $R, C > 0$ et un polynôme P de degré inférieur à u tels que

$$\forall r \leqslant R, \quad \left(\frac{1}{r^d}\int_{B(x_0,r)} \left|f(x) - P(x-x_0)\right|^p \,\mathrm{d}x\right)^{1/p} \leqslant Cr^u.$$

En tenant compte du fait que les extensions « naturelles » (du point de vue de l'analyse harmonique) des espaces L^p pour $p = \infty$ et $p \leqslant 1$ sont, respectivement, l'espace BMO et les espaces de Hardy réels H^p, on peut étendre les conditions $T^p_u(x_0)$ à ces valeurs de p.

Définition 0.2. Soit $p \in (0,1]$, $f \in H^p_{\text{loc}}$ et $u \geqslant -d/p$; $f \in T^p_u(x_0)$ s'il existe $R, C > 0$ et un polynôme P de degré inférieur à u tels que $\|(f-P)1_{B(x_0,r)}\|_p \leqslant Cr^{u+d/p}$.

Soit $f \in BMO_{\text{loc}}$; $f \in T^\infty_u(x_0)$ s'il existe $R, C > 0$ et un polynôme P de degré inférieur à u tels que $\|(f-P)1_{B(x_0,r)}\|_{BMO} \leqslant Cr^u$.

Notre but est d'obtenir une caractérisation de ces conditions pour tout $p \in (0,+\infty]$ par un critère portant sur les modules des coefficients d'ondelette de f.

Soient $\psi^{(i)}$, $i = 1,\ldots,2^d-1$, des fonctions C^A à support compact (où A est choisi suffisamment grand) et engendrant une base d'ondelettes, c'est-à-dire que les $2^{dj/2}\psi^{(i)}(2^jx-k)$ ($i = 1,\ldots,2^d-1$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$) forment une base orthonormée de $L^2(\mathbb{R}^d)$. On utilise l'indexation des ondelettes par les cubes dyadiques (rappelée dans le texte anglais). On notera $c_\lambda = 2^{dj}\int \psi^{(i)}(2^jx-k)f(x)\,\mathrm{d}x$. Si $x_0 \in \mathbb{R}^d$, on notera $\lambda_j(x_0)$ le cube dyadique de largeur 2^{-j} contenant x_0, et $S_f(j,x_0)(x) = (\sum_{\lambda\subset 3\lambda_j(x_0)} |c_\lambda|^2 1_\lambda(x))^{1/2}$.

Théorème 0.3. *Soit $p \in (0,\infty)$ and $u > -d/p$; si $f \in T^p_u(x_0)$, alors $\exists C \geqslant 0$ tel que $\forall j \geqslant 0$,*

$$\left\|S_f(j,x_0)\right\|_p \leqslant C2^{-j(u+d/p)}. \tag{2}$$

Si $p = +\infty$, cette condition devient

$$\forall \lambda \subset 3\lambda_j(x_0), \quad \left(\sum_{\lambda'\subset\lambda} 2^{-dj'}|c_{\lambda'}|^2\right)^{1/2} \leqslant C2^{-dl/2}2^{-uj}, \tag{3}$$

où la largeur de λ est notée 2^{-l} et la largeur de λ' est notée $2^{-j'}$.

Réciproquement, si (2) *est vérifiée* (*ou si* (3) *est vérifiée dans le cas $p = +\infty$*), *et si $u \notin \mathbb{N}$, alors $f \in T^p_u(x_0)$.*

On remarquera que, si $p = 2$, cette caractérisation se simplifie en

$$\sum_{\lambda'\subset 3\lambda_j(x_0)} 2^{-d(j'-j)}|c_{\lambda'}|^2 \leqslant C2^{-2uj},$$

qui avait été obtenue antérieurement par Yves Meyer (communication personelle).

1. Introduction

Several definitions for the pointwise regularity of a function $f:\mathbb{R}^d \to \mathbb{R}$ can be introduced depending on the global assumptions that are made on f. The most widely used is the *Hölder criterium*.

Definition 1.1. Let $f \in L^\infty_{\mathrm{loc}}$, $x_0 \in \mathbb{R}^d$ and $\alpha \geqslant 0$; then $f \in C^\alpha(x_0)$ if $\exists R > 0$, $C > 0$, and a polynomial P of degree less that α such that

$$\text{if } |x - x_0| \leqslant R \text{ then } \left| f(x) - P(x - x_0) \right| \leqslant C|x - x_0|^\alpha. \tag{4}$$

The global setting supplied by L^∞_{loc} is implicitly required by (4); indeed, (4) implies that f is bounded in a neighbourhood of x_0. Thus Definition 1.1 cannot supply a sensible notion of pointwise regularity if the natural setting for f is L^p_{loc} for $p < \infty$. Another drawback is less obvious and was already pointed out by Calderón and Zygmund in 1961, see [2]: The pointwise Hölder condition is not preserved under classical pseudodifferential operators of order 0. This instability property has a counterpart in wavelet analysis: Wavelet bases are not unconditional bases of the space $C^\alpha(x_0)$; even more is true: There exist two functions f and g which share the same moduli of wavelet coefficients, and nonetheless satisfy $f \in C^\alpha(x_0)$ whereas $\forall \beta > 0$, $g \notin C^\beta(x_0)$. Thus Definition 1.1 is unsuitable in several settings; Calderón and Zygmund introduced the following extension which makes sense in the L^p setting and is preserved under singular integral operators.

Definition 1.2. Let $p \in (1, \infty)$, $f \in L^p_{\mathrm{loc}}$ and $u \geqslant -d/p$; then $f \in T^p_u(x_0)$ if $\exists R, C > 0$ and a polynomial P of degree less than u such that

$$\forall r \leqslant R, \quad \left(\frac{1}{r^d} \int_{B(x_0,r)} |f(x) - P(x - x_0)|^p \,\mathrm{d}x \right)^{1/p} \leqslant C r^u. \tag{5}$$

Note that this condition can be rewritten $\|(f - P)1_{B(x_0,r)}\|_p \leqslant C r^{u+d/p}$. If one keeps in mind the requirement of using a criterium which is invariant under pseudodifferential operators of order 0, the following definition is the natural extension of $T^p_u(x_0)$ outside the range $p \in (1, \infty)$. (Recall that, if $p \leqslant 1$, then H^p denotes the real Hardy space, see [6].)

Definition 1.3. Let $p \in (0, 1]$, $f \in H^p_{\mathrm{loc}}$ and $u \geqslant -d/p$; then $f \in T^p_u(x_0)$ if $\exists R, C > 0$ and a polynomial P of degree less that u such that $\|(f - P)1_{B(x_0,r)}\|_p \leqslant C r^{u+d/p}$.

Let $f \in BMO_{\mathrm{loc}}$; then $f \in T^\infty_u(x_0)$ if $\exists R, C > 0$ and a polynomial P of degree less that u such that $\|(f - P)1_{B(x_0,r)}\|_{BMO} \leqslant C r^u$.

Let $p \in (0, +\infty]$; then the p-exponent of f at x_0 is $h^p_f(x_0) = \sup\{u\colon\ f \in T^p_u(x_0)\}$.

The motivations for considering this new types of pointwise conditions are of a different nature for $p = \infty$ and for $p \leqslant 1$. If $p = +\infty$, then the $T^\infty_u(x_0)$ condition is the sharpest condition which is implied by $C^u(x_0)$ and can be characterized by a condition bearing on the moduli of the wavelet coefficients of f; it is therefore stronger than the two-microlocal conditions $f \in C^{u,-u}(x_0)$ of [3]. In particular, while the two-microlocal condition can be satisfied by distributions which do not coincide with a function in a neighbourhood of x_0, the $T^\infty_u(x_0)$ wavelet characterization implies that (5) holds for any $p < \infty$. Another motivation is supplied by the analysis of domains with fractal boundaries; one way to understand the geometry of a domain is to use analytic tools on its characteristic function and, in particular, perform its multifractal analysis. This cannot be done using the Hölder exponent as a measure for pointwise regularity since the Hölder exponent of a characteristic function only takes the two values 0 and $+\infty$; thus no characteristic function is multifractal in this sense. By contrast, the p-exponent can take any non-negative value, thus opening the way to a multifractal analysis of domains, see [5].

If $p < 1$, the condition $f \in H^p_{\mathrm{loc}}$, allows one to deal with singularities such as $|x - x_0|^{-a}$ near x_0 for $a < d/p$; therefore using arbitrarily small values of p allows one to deal with singularities of arbitrarily large exponent a, which is needed in some applications, see [1].

Clearly, $C^u(x_0) \hookrightarrow T^\infty_u(x_0)$ and, if $+\infty \geqslant p \geqslant q > 0$, then $T^p_u(x_0) \hookrightarrow T^q_u(x_0)$. Using the classical interpolation results between L^p and/or H^p spaces, it follows that, if $f \in T^p_u(x_0) \cap T^q_v(x_0)$, and if r is such that $\frac{1}{r} = \frac{\alpha}{r} + \frac{1-\alpha}{q}$ with $0 < \alpha < 1$, then $f \in T^w_r(x_0)$ with $w = \alpha u + (1-\alpha)v$. Thus, for x_0 given, the function $q \to h^{1/q}_f(x_0)$ is defined on an interval of the form $[q_0, +\infty)$ or $(q_0, +\infty)$, where it is concave and increasing.

2. Wavelet characterization

Let $\psi^{(i)}$, $i = 1, \ldots, 2^d - 1$, be compactly supported C^A functions (where A is large enough) generating a wavelet basis, i.e. the $2^{dj/2}\psi^{(i)}(2^j x - k)$ ($i = 1, \ldots, 2^d - 1$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$) form an orthonormal basis of $L^2(\mathbb{R}^d)$. Wavelets will be indexed by dyadic cubes as follows: We can consider that i takes values among all dyadic subcubes λ_i of $[0,1)^d$ of width $1/2$ except for $[0,1/2)^d$; thus, the set of indices (i, j, k) can be relabelled using dyadic cubes: λ denotes the cube $\{x\colon 2^j x - k \in \lambda_i\}$; we note $\psi_\lambda(x) = \psi^{(i)}(2^j x - k)$ (an L^∞ normalization is used), and $c_\lambda = 2^{dj} \int \psi_\lambda(x) f(x)\,\mathrm{d}x$; $\exists C\colon \operatorname{supp}(\psi_\lambda) \subset C\lambda$ where $C\lambda$ denotes the cube of same center as λ and C times wider. If $x_0 \in \mathbb{R}^d$, then $\lambda_j(x_0)$ denotes the unique dyadic cube of width 2^{-j} which contains x_0, and the *local square function* is $S_f(j, x_0)(x) = (\sum_{\lambda \subset 3\lambda_j(x_0)} |c_\lambda|^2 1_\lambda(x))^{1/2}$.

Theorem 2.1. *Let $p \in (0, \infty)$ and $u > -d/p$; if $f \in T^p_u(x_0)$, then $\exists C \geqslant 0$ such that $\forall j \geqslant 0$,*

$$\left\| S_f(j, x_0) \right\|_p \leqslant C 2^{-j(u+d/p)}. \tag{6}$$

If $p = +\infty$, this condition becomes

$$\forall \lambda \subset 3\lambda_j(x_0), \quad \left(\sum_{\lambda' \subset \lambda} 2^{-dj'} |c_{\lambda'}|^2 \right)^{1/2} \leqslant C 2^{-dl/2} 2^{-uj}, \tag{7}$$

where 2^{-l} is the width of λ and $2^{-j'}$ is the width of λ'.

Conversely, if (6) *holds* (*or if* (7) *holds in the case $p = +\infty$*) *and if $u \notin \mathbb{N}$, then $f \in T^p_u(x_0)$.*

Proof of Theorem 2.1. Assume first that $p < \infty$; then (see [6]) $f \in L^p(\mathbb{R}^d)$ if $p > 1$, or $f \in H^p(\mathbb{R}^d)$ if $p \leqslant 1$ if and only if $(\sum_\lambda |c_\lambda|^2 1_\lambda(x))^{1/2} \in L^p$. The direct part of the theorem follows by applying this characterization to $g(x) = (f(x) - P(x - x_0)) 1_{B(x_0, D2^{-j})}(x)$ and noticing that, if D is large enough and $\lambda \subset 3\lambda_j(x_0)$, then the corresponding wavelet coefficients of f and g coincide. If $p = +\infty$, the argument is the same using the characterization of BMO, see [6]: $\exists C, \forall \lambda, \sum_{\lambda' \subset \lambda} 2^{-dj'} |c_{\lambda'}|^2 \leqslant C \mathrm{Meas}(\lambda)$.

Let us now prove the converse part. We can forget the 'low frequency component' of f corresponding to $j < 0$ in its wavelet decomposition, since its contribution belongs locally to $C^A(\mathbb{R}^d)$. Let Λ_j denote the set of dyadic cubes of width 2^{-j}, $\Delta_j f = \sum_{\lambda \in \Lambda_j} c_\lambda \psi_\lambda$, and let $P_j(x - x_0)$ denote the Taylor polynomial of $\Delta_j f$ of degree $[u]$ at x_0; (6) or (7) imply that,

$$\text{if } \operatorname{dist}(\lambda, x_0) \leqslant D 2^{-j}, \text{ then } |c_\lambda| \leqslant C 2^{-uj}. \tag{8}$$

Let $\rho > 0$ be fixed and let J be defined by $2^{-J} \leqslant \rho < 2 \cdot 2^{-J}$ and L be a constant which will be fixed later, but depends only on the size of the support of the wavelets. If $j \leqslant J + L$, then at most C of the ψ_λ have a support intersecting $B = B(x_0, \rho)$ and each of them satisfies (8). It follows from Taylor's formula that, if $x \in B$ and $j \leqslant J + L$, then $|\Delta_j f(x) - P_j(x - x_0)| \leqslant C \rho^{[u]+1} 2^{j([u]+1-u)}$, and therefore

$$\sum_{j=0}^{J+L} \left| \Delta_j f(x) - P_j(x - x_0) \right| \leqslant C \rho^u. \tag{9}$$

It follows also from (8) that, if $|k| \leqslant [u]+1$, then, $\forall j \geqslant 0$, $|\Delta_j^{(k)} f(x_0)| \leqslant 2^{(|kl-u)j}$; therefore the series $P(x-x_0) = \sum_{j=0}^{\infty} P_j(x-x_0) = \sum_{j=0}^{\infty} \sum_{|k|<u} \frac{\Delta_j f^{(k)}(x_0)}{k!}(x-x_0)^k$ converges and, if $|x-x_0| \leqslant \rho$, then

$$\sum_{j=J+L}^{\infty} \left|P_j(x-x_0)\right| \leqslant C \sum_{j=J+L}^{\infty} \sum_{|k|<u} 2^{(|k|-u)j} \rho^k \leqslant C\rho^u. \tag{10}$$

Let now $g_J(x) = \sum_{j=J+L}^{\infty} \Delta_j f(x)$; then $\|g_J 1_B\|_p \leqslant \| \sum_{j=J+L}^{\infty} \sum_{\lambda \subset B(x_0,2\rho)} c_\lambda \psi_\lambda \|_p$ where L has been picked large enough so that both functions coincide on B. Using the wavelet characterization of L^p, the right hand side is bounded by

$$C \left\| \left(\sum_{j=J+L}^{\infty} \sum_{\lambda \subset B(x_0,2\rho)} |c_\lambda|^2 1_\lambda \right)^{1/2} \right\|_p \leqslant S_f(j-L, x_0) \leqslant C 2^{-j(u+d/p)}. \tag{11}$$

The required estimate for $\|(f - P(x-x_0))1_{B(x_0,\rho)}\|_p$ follows immediately from (9), (10) and (11).

The case $p = \infty$ is completely similar.

3. Remarks and implications in multifractal analysis

If $p = 2$, this characterization boils down to a local l^2 condition on the wavelet coefficients

$$\sum_{\lambda' \subset 3\lambda_j(x_0)} 2^{-d(j'-j)} |c_{\lambda'}|^2 \leqslant C 2^{-2uj} \tag{12}$$

which was previously obtained by Yves Meyer (personal communication) using an alternative proof.

If $p = +\infty$, and if $1 \leqslant p < +\infty$, then Theorem 2.1 improves previous results of, respectively, [3] and [5]; up to now, the converse part required a uniform regularity assumption $f \in B_p^{\epsilon,p}$ for and $\epsilon > 0$, which turns out to be unnecessary. Note also that, if f satisfies (7), then $f \in T_u^p(x_0)$ for any $p < \infty$. This is in sharp contrast with the two-microlocal condition obtained in [3] as a consequence of $C^u(x_0)$ regularity which does not imply any $T_u^p(x_0)$ regularity result (or even that f locally coincides with a function).

If $p \neq 2$, then (6) in not a local l^p condition on the wavelet coefficients; however, the embeddings between Sobolev and Besov spaces supply the following conditions which are easier to use in practice:

If $p \geqslant 2$, then $L^p \hookrightarrow B_p^{0,p}$; thus if $f \in T_u^p(x_0)$ for $p \geqslant 2$, then $\sum_{\lambda' \subset 3\lambda_j(x_0)} 2^{-d(j'-j)} |c_{\lambda'}|^p \leqslant C 2^{-puj}$. Similarly, if $p \leqslant 2$, then $B_p^{0,p} \hookrightarrow L^p$; thus if $\sum_{\lambda' \subset 3\lambda_j(x_0)} 2^{-d(j'-j)} |c_{\lambda'}|^p \leqslant C 2^{-puj}$, then $f \in T_u^p(x_0)$.

The two-microlocal condition $C^{\alpha,-\alpha}(x_0)$ is 'far' from the Hölder condition $C^\alpha(x_0)$ in the sense that it can be satisfied by distributions which are not functions. However, it is 'close' if a uniform regularity condition holds; indeed, let $\alpha > \epsilon > 0$; if $f \in C^{\alpha,-\alpha}(x_0) \cap C^\epsilon(\mathbb{R}^d)$, then $\forall \beta < \alpha$, $f \in C^\beta(x_0)$, see [3]. The following result shows that T_u^p regularity is 'farther' from Hölder regularity under this respect.

Proposition 3.1. *Let $f \in T_\alpha^p(x_0) \cap C^\epsilon(\mathbb{R}^d)$, with $\alpha > \epsilon + (d/p)$ and let $\beta = \alpha\epsilon p/(\epsilon p + d)$; then $f \in C^\beta(x_0)$ and this result is optimal.*

The proof is similar to the proof of the converse part in Theorem 2.1; the only difference lies in the estimate of $\sup|\Delta_j f|$ on $B = B(x_0, 2^{-J})$ for $j \geqslant J$. The uniform regularity assumption implies that $\sup(|\Delta_j f| \leqslant C 2^{-\epsilon j}$; the $T_\alpha^p(x_0)$ assumption implies that $\|\Delta_j f\|_{L^p(B)} \leqslant C 2^{-\alpha J}$, which, using Bernstein's inequalities, implies that $\sup|\Delta_j f| \leqslant C 2^{-\alpha J} 2^{dj/p}$. The conclusion follows by picking the best of these two estimates according to the value of j.

The purpose of the multifractal analysis of a function f is to determine the Hausdorff dimensions of the sets of points where f has a given pointwise regularity. Up to now, this was performed mainly in the context of Hölder pointwise regularity. We now give a result for T_u^p regularity. In that case one wishes to determine the *p-spectrum* $D_p^f(H) = \dim(\{x\colon h_f^p(x) = H\})$ (where dim denotes the Hausdorff dimension). Upper bounds on the p-spectrum can be derived in terms of the following quantities. Let

$$S_f^\lambda(p) = \left(\int_\lambda \left(\sum_{\lambda' \subset \lambda} |c_{\lambda'}|^2 1_{\lambda'}(x) \right)^{p/2} \mathrm{d}x \right)^{1/p},$$

$$\eta_f^p(q) = \lim_{R\to+\infty} \liminf_{j\to+\infty} \frac{\log(2^{d(q/p-1)j} \sum_{\lambda\in\Lambda_j \cap B(0,R)} (S_f^\lambda(p))^q)}{\log(2^{-j})}.$$

Theorem 3.2. *Let $f \in L^p_{\mathrm{loc}}$; then $D_p^f(H) \leqslant \inf_{q\neq 0}(d - \eta_f^p(q) + Hq)$.*

Sketch of proof. It follows from Theorem 2.1 that

$$h_f^p(x_0) = -\frac{d}{p} + \liminf_{j\to+\infty} \left(\frac{-1}{j} \log_2 \left(\sup_{\lambda'\in \mathrm{adj}(\lambda)} S_f^{\lambda'}(p) \right) \right) \tag{13}$$

where $\mathrm{adj}(\lambda)$ denotes the 3^d dyadic cubes of same width as λ and such that $\bar\lambda \cap \bar\lambda' \neq \emptyset$. The proof of Theorem 3.2 is exactly the same as the upper bound for the Hölder spectrum, see [4], since the only property used in the derivation of this upper bound is a formula similar to (13). Note that, in Theorem 3.2, no global regularity assumption is needed since, in [4], this assumption is needed only to insure the validity of the formula corresponding to (13), but not in the proof of the upper bound.

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