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Probability Theory

Level sets of β-expansions

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Abstract

Let $\{\epsilon_n(x)\}_{n\geqslant 1}$ be the sequence of β-digits of a real number $x \in (0,1)$, with the golden number $\beta = (\sqrt{5}+1)/2$ as basis. For any $0 \leqslant p \leqslant 1/2$, any $0 < \tau \leqslant 1$ and any real number a, we consider the level set consisting of numbers x such that $\sum_{n=1}^{\infty} (\epsilon_n(x) - p)/n^{\tau} = a$. We prove that the Hausdorff dimension of this set is independent of a and τ, and that it is equal to $\log f(p)/\log\beta$ where $f(p) = (1-p)^{1-p}/((1-2p)^{1-2p}p^p)$. ***To cite this article: A. Fan, H. Zhu, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Ensembles de niveau des β-développements. Soit $\{\epsilon_n(x)\}_{n\geqslant 1}$ la suite de β-digits du nombre réel $x \in (0,1)$, avec le nombre d'or $\beta = (\sqrt{5}+1)/2$ comme base. Pour tout $0 \leqslant p \leqslant 1/2$, $0 < \tau \leqslant 1$ et $a \in \mathbb{R}$, nous considérons l'ensemble de niveau qui est constitué des x tels que $\sum_{n=1}^{\infty} (\epsilon_n(x) - p)/n^{\tau} = a$. Nous prouvons que la dimension de Hausdorff de cet ensemble est independante de a et τ, et qu'elle est égale à $\log f(p)/\log\beta$ où $f(p) = (1-p)^{1-p}/((1-2p)^{1-2p}p^p)$. ***Pour citer cet article : A. Fan, H. Zhu, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

1. Introduction

Let $\beta > 1$ be a real number. It is well known that any number $x \in [0,1)$ has a β-expansion $x = \sum_{i=1}^{\infty} \varepsilon_i(x)/\beta^i$ where $\varepsilon_i(x) = [\beta T^{i-1}(x)]$, $T(x) = \beta x \pmod 1$ being the β-shift on $[0,1)$ and $[y]$ denoting the integral part of a real number y (see [8,7]). We call $\{\varepsilon_n(x)\}_{n\geqslant 1}$ the sequence of β-digits of x. In this note we study the distribution of the β-digits for different numbers x when $\beta = (\sqrt{5}+1)/2$ is the golden number.

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Let $S_n(x) = \sum_{j=1}^{n} \epsilon_k(x)$. We introduce the following sets:

$$E(p) = \{x \in [0,1):\ S_n(x) - np = o(n)\} \quad (p \in [0, 1/2]),$$
$$L(p,\tau,a) = \{x \in [0,1):\ \textstyle\sum_{k=1}^{\infty} k^{-\tau}(\epsilon_k(x) - p) = a\} \quad (p \in [0,1/2],\ 0 < \tau \leqslant 1,\ a \in \mathbb{R}),$$

and we consider the Hausdorff dimensions of these sets. It is well known that $\dim E(p) = \log f(p)/\log\beta$ with $f(p) = (1-p)^{1-p}/((1-2p)^{1-2p}p^p)$ (see [3], for example). Observe that the level sets $L(p,\tau,a)$ are disjoint subsets of $E(p)$. However, we prove that they have all the same dimension as $E(p)$.

Theorem 1.1. *We have* $\dim L(p,\tau,a) = \dim E(p)$ *for all* $0 \leqslant p \leqslant 1/2$, $0 < \tau \leqslant 1$ *and* $-\infty < a < +\infty$.

The result is a kind of refinement of Birkhoff ergodic theorem. Another kind of refinement is considered in [2]. The method for proving the above theorem could be adapted for other Pisot numbers $\beta > 1$ than the golden number. For the dyadic expansion (i.e. $\beta = 2$), the function $f(p)$ must be replaced by $p^p(1-p)^{1-p}$ where $0 \leqslant p \leqslant 1$. Wu [9] and Xi [10] studied the dyadic case with $p = 1/2$ (the mean value of $\epsilon_n(x)$ with respect to the Lebesgue measure) and proved that $\dim_H L(1/2,\tau,a) = 1$. Earlier, Beyer [1] showed the inequality $\dim_H L(1/2,\tau,a) \geqslant 1/2$.

Our study gives a very partial contribution to the following general problem. Given any function ϕ, we consider $S_n\phi(x) = \sum_{j=0}^{n-1}\phi(T^j x)$. For any ergodic invariant measure μ, the Birkhoff theorem asserts that $S_n\phi(x) - n\int\phi\,\mathrm{d}\mu = o(n)$ for μ-almost all x. In [2], we have studied possible refinements by considering points x such that $S_n\phi(x) - n\int\phi\,\mathrm{d}\mu \asymp n^\tau$ with $0 < \tau < 1$. Another way to refine the Birkhoff theorem is to consider the set of points such that the series $\sum_{n=1}^{\infty} a_n(\phi(T^j x) - \int\phi\,\mathrm{d}\mu)$ converges, where a_n is a decreasing positive sequence. Our above theorem concerns nothing but the occurrence of digits, for ϕ is the characteristic function of the interval $[0,\beta^{-1}]$. The general case remains unsolved. Another special case is the trigonometric series $\sum_{n=1}^{\infty} a_n(e^{2\pi i 2^n x} - p)$ where p may be complex. It corresponds to $\beta = 2$, $\phi(x) = e^{2\pi i x}$. When $p = 0$ (the mean value of $e^{2\pi i x}$ with respect to the Lebesgue measure), for any complex number a there exists points x such that $\sum_{n=1}^{\infty} a_n(e^{2\pi i 2^n x} - p) = a$ (see [6]). Little is known about the level sets of this series.

2. Preliminaries

Let $a_n = n^{-\tau}$. The sequence $\{a_n\}$ shares the following property, the most useful one to us,

$$\lim_{n\to\infty} a_n = 0, \quad \sum_{n=1}^{\infty}|a_n| = +\infty, \quad \sum_{n=1}^{\infty}|a_n - a_{n+1}| < +\infty. \tag{1}$$

It is known [7] that for the golden number β, the set of sequences of β-digits coincides with the subshift of finite type Σ_A determined by the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, with an exception of a countable set which will be taken off, because the transformation $Tx = \beta x \pmod 1$ is Markovian. Let $\eta : [0,1) \to \Sigma_A$ be the function, which associates x to its β-digits $\{\varepsilon_n(x)\}$, is one-to-one except for a countable set and is strictly increasing when Σ_A is endowed with the lexicographical order.

Any finite or infinite sequence of 0 or 1 which does not contain the string 11 is said to be admissible. For any admissible sequence $\{\epsilon_n\}_{1\leqslant n\leqslant N}$, the β-interval $I(\epsilon_1,\ldots,\epsilon_N)$ is defined to be the set of all $x \in [0,1)$ such that $\epsilon_n(x) = \epsilon_n$ for $1 \leqslant n \leqslant N$. A natural metric on Σ_A is defined by $d(\epsilon,\eta) = \beta^{-n}$ where n is the largest integer such that $\epsilon_i = \eta_i$ for $1 \leqslant i \leqslant n$. The β-interval $I(\epsilon_1,\ldots,\epsilon_n)$ has a length of order β^{-n} (see [4]).

Let $J \geqslant 1$ be a big fixed integer. We define the 'killing map' $\widehat{T} : \Sigma_A \to \Sigma_A$ by

$$\widehat{T}(\varepsilon_1, \varepsilon_2, \ldots) = (\eta_1, \eta_2, \ldots),$$

where $\eta_n = 0$ or ϵ_n according to n is a multiple of J or not. Notice that $\widehat{T}$ is Lipschitzian. Then consider the map $T : [0, 1) \to [0, 1)$ defined by $T = \eta^{-1}\widehat{T}\eta$.

Lemma 2.1. *We have* $\dim_H TE \leqslant \dim_H E$ *for any set* $E \subset [0, 1)$.

Proof. It suffices to notice that both η and η^{-1} preserve the Hausdorff dimension and that $\widehat{T}$ is Lipschitzian. □

Lemma 2.2 (Kaczmarz–Steinhaus [5]). *Suppose that* $\{a_n\}$ *is sequence of real numbers such that* $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} |a_n| = \infty$ *and that* $\{p_n\}$ *and* $\{q_n\}$ *be two sequences of real numbers such that* $-\Delta \leqslant q_n \leqslant -\delta$ *and* $\delta \leqslant p_n \leqslant \Delta$ *for some constants* $\Delta \geqslant \delta > 0$. *Then for any real number* a, *there is a sequence* $\{R_n\}$ *with* $R_n = p_n$ *or* q_n *such that* $\sum_{n=1}^{\infty} a_n R_n = a$.

Proof. We may choose R_n inductively. Suppose that $R_1, \ldots, R_n$ are chosen. We take $R_{n+1}=q_{n+1}$ if $\sum_{j=1}^{n} a_n R_j > a$; otherwise we take $R_{n+1} = p_{n+1}$. □

Lemma 2.3. *Let* $\{\epsilon_k\}_{k\geqslant 1} \in \{0, 1\}^{\mathbb{N}}$ *and* $\{a_k\}_{k\geqslant 1} \in \mathbb{R}^{\mathbb{N}}$. *For any integers* $n < m$, *denoting* $\mu = \frac{1}{m-n}\sum_{j=n+1}^{m} \epsilon_j$ (*i.e. the frequency of* 1) *we have*

$$\left|\sum_{k=n+1}^{m} a_k(\epsilon_k - \mu)\right| \leqslant \sum_{j=n+1}^{m} \epsilon_j \sum_{k=n+1}^{m} |a_k - a_{k-1}|.$$

Proof. Let $N = \sum_{j=n+1}^{m} \epsilon_j$. We may write

$$\sum_{k=n+1}^{m} a_k(\epsilon_k - \mu) = \frac{1}{n-m}\left[\sum_{k:\ \epsilon_k=1} (n-m)a_k - \sum_{k=n+1}^{m} Na_k\right].$$

Both sums at the right-hand side may be considered as sums of a_i's with $N(n-m)$ terms. Notice that for any a_i and a_j with $n < i < j \leqslant m$ we have

$$|a_i - a_j| \leqslant \sum_{k=i+1}^{j} |a_k - a_{k-1}| \leqslant \sum_{k=n+1}^{m} |a_k - a_{k-1}|.$$

So, $|\sum_{k=n+1}^{m} a_k(\epsilon_k - \mu)| \leqslant N\sum_{k=n+1}^{m} |a_k - a_{k-1}|$. □

3. Proof

We have only to prove $\dim L(p, \tau, a) \geqslant \log f(p)/\log\beta$ for $0 < p < 1/2$. Take an infinite number of couples of integers (J, W) such that $W/(J-1) < p < (W+1)/(J-1)$. For such a fixed couple (J, W), we construct a set $F_J \subset [0, 1]$ as follows. Let G'_J be the set of the β-admissible sequences $\{\varepsilon_n\}_{1\leqslant n\leqslant J}$ of length J such that (i) $\varepsilon_1 = 0$, $\varepsilon_{J-1} = \varepsilon_J = 0$; (ii) $\sum_{i=1}^{J-1} \varepsilon_i = W$. Let G''_J be the set of the β-admissible sequences $\{\varepsilon_n\}_{1\leqslant n\leqslant J}$ of length J such that (iii) $\varepsilon_1 = 0$, $\varepsilon_{J-1} = \varepsilon_J = 0$; (iv) $\sum_{i=1}^{J-1} \varepsilon_i = W + 1$. For any $t \geqslant 1$, let $\Lambda_t = [J(t-1)+1, Jt-1] \cap \mathbb{N}$ and $A_t = \sum_{i\in\Lambda_t} a_i$. We have $\sum_{t=1}^{\infty} |A_t| = \infty$ and $\lim_{t\to\infty} A_t = 0$. Notice that $W/(J-1) - p < 0$ and $(W+1)/(J-1) - p > 0$. By Lemma 2.2, for any $\alpha \in \mathbb{R}$ we can find a sequence $\{r_t\}_{t\geqslant 1}$ with $r_t = W/(J-1) - p$ or $(W+1)/(J-1) - p$ such that

$$\sum_{t=1}^{\infty} A_t r_t = \alpha. \tag{2}$$

Define $G_t = G'_J$ or G''_J according to $r_t = W/(J-1) - p$ or $(W+1)/(J-1) - p$. Then define $G = \prod_{t=1}^{\infty} G_t$ and F_J to be the set of all $x = \sum_{n=1}^{\infty} \varepsilon_n/\beta^n$ with $\{\epsilon_n\}_{n\geqslant 1} \in G$. Now for $\{\epsilon_n\}_{n\geqslant 1} \in G$, we are going to show the convergence of the series $\sum_{i\in\mathbb{N}\setminus J\mathbb{N}}^{\infty} a_i(\varepsilon_i - p)$. Let $B_t = \sum_{i\in\Lambda_t} a_i(\varepsilon_i - W/(J-1))$ or $\sum_{i\in\Lambda_t} a_i(\varepsilon_i - (W+1)/(J-1))$ according to $r_t = W/(J-1) - p$ or $(W+1)/(J-1) - p$. Then we have $\sum_{i\in\Lambda_t} a_i(\varepsilon_i - p) = B_t + A_t r_t$. By Lemma 2.3, we have $|B_t| < (W+1)\sum_{i\in\Lambda_t} |a_{i+1} - a_i|$. It follows that $\sum_{t=1}^{\infty} |B_t| < +\infty$. Thus $\sum_{t=1}^{\infty} B_t$ is convergent. We denote its sum by γ. This convergence, together with (2), implies

$$\sum_{i=1\in\mathbb{N}\setminus J\mathbb{N}} a_i(\varepsilon_i - p) = \sum_{t=1}^{\infty}\sum_{i\in\Lambda_t} a_i(\varepsilon_i - p) = \gamma + \alpha. \tag{3}$$

According to Lemma 2.2, we can find a new sequence $\{\varepsilon'_{Ji}\}$ taking in $\{0, 1\}$ such that

$$\sum_{i=1}^{\infty} a_{Ji}(\varepsilon'_{Ji} - p) = a - (\gamma + \alpha). \tag{4}$$

Let $E_J = \eta^{-1}\widetilde{T}\eta(F_J)$. By (3) and (4), we get $E_J \subseteq L_a$ then $F_J \subset TL_a$.

By Lemma 2.1, we have to estimate $\dim F_J$ from below. For $1 \leqslant i_t \leqslant \operatorname{Card} G_t$ $(1 \leqslant t \leqslant n)$, let $U_{i_1 i_2\cdots i_n} = [x_{i_1 i_2\cdots i_n}, x_{i_1 i_2\cdots i_n} + \beta^{-Jn}]$ where $x_{i_1 i_2\cdots i_n} = \sum_{k=1}^{Jn} \varepsilon_k/\beta^k$ with $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{Jn}) \in \prod_{t=1}^{n} G_t$. The interval $U_{i_1\cdots i_n}$ is nothing but the β-interval $I(\epsilon_1, \ldots, \epsilon_{Jn})$. Since $\varepsilon_{Jn-1} = 0$, all these intervals $U_{i_1 i_2\cdots i_n}$ are disjoint. We have the expression $F_J = \bigcap_{n=1}^{\infty} \bigcup_{i_1 i_2\cdots i_n} U_{i_1 i_2\cdots i_n}$. Define the set function μ by

$$\mu(U_{i_1 i_2\cdots i_n}) = \frac{1}{(\operatorname{Card} G'_J)^{u_n} (\operatorname{Card} G''_J)^{v_n}},$$

where u_n is the number of G'_J's in the sequence $\{G_1, \ldots, G_n\}$ and $v_n = n - u_n$. We can extend μ to a Borel probability measure on F_J. Write $\mu(U_{i_1 i_2\cdots i_n}) = |U_{i_1 i_2\cdots i_n}|^{s_n}$ where $s_n = (u_n \log\operatorname{Card} G'_J + v_n \log\operatorname{Card} G''_J)/(nJ\log\beta)$. Without loss of generality, we assume $\operatorname{Card} G'_J \geqslant \operatorname{Card} G''_J$. Then

$$\mu(U_{i_1 i_2\cdots i_n}) \leqslant |U_{i_1 i_2\cdots i_n}|^{\frac{\log\operatorname{Card} G'_J}{J\log\beta}}.$$

This inequality remains true for general intervals instead of $U_{i_1 i_2\cdots i_n}$ because the lengths of intervals $U_{i_1 i_2\cdots i_n}$ (n being fixed) are between $c_1\beta^{-Jn}$ and $c_2\beta^{-Jn}$ for some constants $0 < c_1 \leqslant c_2$. Then by the Frostman lemma, we get $\dim_H F_J \geqslant (\log\operatorname{Card} G'_J)/(J\log\beta)$. Notice that $\operatorname{Card} G'_J = \binom{J-3-W}{W}$ is a combinatorial number; it is easy to compute $\lim_J (\log\operatorname{Card} G'_J)/J = \log f(p)$. Since $L_a \supset E_J$, we have proved the theorem.

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