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Numerical Analysis

Domain decomposition methods of dual-primal FETI type for edge element approximations in three dimensions

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Abstract

We consider domain decomposition algorithms of FETI type for edge element approximations in three dimensions. We first show that a strong coupling exists between tangential degrees of freedom associated to the subdomain edges and faces. We then propose a dual-primal FETI algorithm that relies on a change of basis and on a suitable choice of a coarse space. We give a logarithmic bound for the condition number of the resulting preconditioned operator. Numerical results confirm this bound and the necessity of performing a change of basis. ***To cite this article: A. Toselli, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Méthodes FETI dual-primal pour approximations aux éléments finis d'arête en dimension trois. Nous considérons des algorithmes FETI pour des approximations en éléments finis d'arête en dimension trois. Nous montrons d'abord qu'il existe un couplage fort entre les degrés de liberté tangentiels associés aux arêtes et aux faces des sous-domaines. Nous proposons ensuite un algorithme FETI dual-primal qui utilise un changement de base et un choix particulier pour le solveur grossier. Nous donnons une borne logarithmique pour le nombre de conditionnement de l'algorithme. Les tests numériques confirment cette borne et la nécessité du changement de base. ***Pour citer cet article : A. Toselli, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Nous considérons le problème (1) en dimension trois et l'espace $H(\mathbf{curl};\Omega)$. On introduit un maillage régulier $\mathcal{T}_h$ de Ω et une partition $\mathcal{T}_H$, sans recouvrement, en sous-domaines Ω_i. Chaque sous-domaine (sous-structure) est l'union d'éléments fins de $\mathcal{T}_h$. Les diamètres de $\mathcal{T}_h$ et $\mathcal{T}_H$ sont h et H, respectivement. On choisit les espaces d'éléments finis d'arête X_h d'ordre le plus bas, introduits dans [6]. Les degrés de liberté peuvent être choisis comme

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a composante tangentielle sur les arêtes de $\mathcal{T}_h$ et, sur un sous-domaine Ω_i peuvent être partagés entre trois classes : *intérieur*, *arête*, et *face* (voir la Fig. 1).

Les algorithmes FETI sont des méthodes de décomposition de domaine sans recouvrement pour la résolution de systèmes algébriques provenants de l'approximation des équations aux dérivées partielles. Ils sont couramment utilisés pour des problèmes de très grande taille, en élasticité et en mécanique des fluides, mais le développement d'une stratégie robuste pour les problèmes électromagnétiques tridimensionnels manque à ce jour.

Les méthodes de décomposition de domaine sans recouvrement utilisent la possibilité de trouver des décompositions stables pour les fonctions d'éléments finis en composantes associées aux faces et aux arêtes des sous-domaines ; voir [2, Section 5]. Ces décompositions existent pour les éléments nodaux, mais ne sont pas disponibles pour les éléments d'arête en dimension trois. Cela est montré Fig. 1. On considère le gradient d'une fonction nodale continue, nulle sur tous les noeuds de Ω_i sauf en un noeud situé sur une arête E. Le rotationnel de ce vecteur est nul et sa norme $H(\mathbf{curl};\Omega_i)$ est $\mathrm{O}(h)$. Si on découple ce vecteur en composantes associées à l'arête E et aux deux faces de E, on obtient des vecteurs avec un rotationnel non nul et d'une norme $H(\mathbf{curl};\Omega_i)$ qui est $\mathrm{O}(h^{-1})$; voir la Section 2.

Nous considérons donc un changement de base. Seules les fonctions de base associées aux arêtes des sous-domaines sont changées ; voir la Définition 3.1. Avec cette nouvelle base on introduit une méthode de type FETI dual-primal ; voir [3,1,7]. On utilise un espace $\widetilde{X}_h$ d'éléments finis discontinus à travers les sous-domaines et on impose la continuité de la composante tangentielle avec des multiplicateurs de Lagrange λ ; voir l'Éq. (3). La méthode FETI utilise la résolution de l'équation pour λ par une méthode itérative comme le gradient conjugué et un préconditionneur M^{-1} ; voir l'Éq. (4).

La forme du préconditionneur M^{-1} est standard et peut être trouvée en [3,4] ou [7, Section 3]. Il reste seulement à caractériser l'espace $\widetilde{X}_h$. Les vecteurs de $\widetilde{X}_h$ satisfont quelques conditions de continuité (*primal constraints*) à travers les sous-domaines, ces conditions déterminent la taille du solveur grossier. Ici on impose la continuité de la moyenne (moment d'ordre zéro) et du moment du premier ordre de la composante tangentielle des vecteurs de $\widetilde{X}_h$ le long des arêtes E des sous-domaines. On obtient donc deux conditions pour chaque arête E.

On trouve que le nombre de conditionnement de l'operateur préconditionné est borné par une fonction logarithmique de H/h et est indépendant du nombre des sous-domaines et des sauts d'un des coefficients ; voir le Théorème 5.1.

Le Tableau 1 montre les nombres de conditionnement pour la méthode FETI avec (à gauche) et sans (à droite) changement de base décrit dans la Section 3. Il s'agit ici de maillages et de partitions uniformes pour des coefficients constants. Les résultats de gauche montrent des nombres de conditionnement très petits, indépendants de la taille n^2 du problème quand H/h est donné. Les résultats de droite, par contre, montrent des nombres de conditionnement très grands qui croissent comme $n^2 = h^{-2}$ confirmant donc notre analyse présentée dans la Section 2.

1. Introduction

We consider the boundary value problem:

$$\begin{aligned} &\mathbf{curl}(a\,\mathbf{curl}\,\mathbf{u}) + b\mathbf{u} = \mathbf{f} \quad \text{in } \Omega,\\ &\mathbf{n}\times(\mathbf{u}\times\mathbf{n}) = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

with Ω a bounded polyhedral domain in $\mathbb{R}^3$ with outward unit normal $\mathbf{n}$ and a and b positive functions. More general boundary conditions can also be considered.

The variational formulation and the analysis of problem (1) require the Sobolev space $H(\mathbf{curl};\Omega)$ of vectors in $L^2(\Omega)^3$ (the space of square summable vectors on Ω), with a **curl** that is also in $L^2(\Omega)^3$. For $\mathbf{u}\in H(\mathbf{curl};\Omega)$, we use the norm given by

$$\|\mathbf{u}\|^2_{H(\mathbf{curl};\Omega)} = \|\mathbf{u}\|^2_{L^2(\Omega)^3} + \|\,\mathbf{curl}\,\mathbf{u}\|^2_{L^2(\Omega)^3},$$

with $\|\cdot\|^2_{L^2(\Omega)^3}$ the norm in $L^2(\Omega)^3$. See, e.g., [5] for an introduction and for further details.

We next consider a conforming triangulation $\mathcal{T}_h$ of Ω, consisting of affinely mapped cubes or tetrahedra, and a nonoverlapping subdomain partition $\mathcal{T}_H = \{\Omega_i\}$, with subdomains that are collections of fine elements. The subdomains (substructures) and the fine elements are always assumed to be shape-regular and H and h, respectively, denote the maximum of their diameters.

We choose the lowest-order edge element (Nédélec) finite element spaces $X_h \subset H(\mathbf{curl}; \Omega)$ on the fine triangulation $\mathcal{T}_h$; see [6]. We also refer to [5] for a fine presentation. Here, we recall that these finite elements preserve the continuity of the tangential component across the element boundaries and the degrees of freedom can be chosen as the (constant) tangential component along the element edges of $\mathcal{T}_h$.

We note that in a subdomain Ω_i, the degrees of freedom can be partitioned into three classes: *interior*, *edge*, and *face*, according to whether they lie in the interior of Ω_i, on a subdomain edge E, or face F. We refer to Fig. 1, left, for the case of a cubical substructure.

The finite element approximation of problem (1) gives rise to a positive definite, symmetric linear system, for the vector u of degrees of freedom:

$$Ku = f. \tag{2}$$

2. Nonoverlapping algorithms for edge elements in three dimensions

FETI algorithms are particular domain decomposition methods of nonoverlapping type for the solution of algebraic systems arising from the approximation of a partial differential equation (see Eq. (2)): they rely on a nonoverlapping partition of Ω into subdomains or substructures. While they are now routinely employed for the solution of huge elasticity and flow problems (see, e.g., [3,1,4]) a full understanding of robust and efficient strategies for nonoverlapping domain decomposition preconditioners for three-dimensional electromagnetic problems is still missing.

Nonoverlapping domain decomposition preconditioners rely on decoupling degrees of freedom associated to geometrical objects associated to subdomains, typically vertices, edges, and faces for three-dimensional continuous nodal elements; see, e.g., [2, Section 5]. For edge elements, we only need to consider subdomain edges and faces. The performance of the corresponding preconditioned iterative method depends on how weak the coupling between the different blocks of degrees of freedom is. This decoupling may appear explicitly in the construction of finite element subspaces as in wire basket methods, [2], but it may also be hidden in the algorithm and may not appear explicitly in the subspaces considered, as in FETI methods.

Decompositions into edge and face components are fairly harmless (i.e., logarithmically stable) operations for continuous nodal h finite elements, see [2], but turn out to be disastrous for edge element approximations. More precisely, we refer to Fig. 1, right, and consider the gradient of a continuous, scalar, piecewise trilinear function ϕ_E with vanishing nodal values on the closure of a subdomain Ω_i except at one node on a coarse edge E where it is one. Since ϕ_E decreases linearly from one to zero along an edge of length $\mathrm{O}(h)$, its tangential component is $\mathrm{O}(h^{-1})$. This vector is curl free and has a low energy:

$$\|\nabla\phi_E\|^2_{H(\mathbf{curl};\Omega_i)} = \|\nabla\phi_E\|^2_{L^2(\Omega_i)^3} = \mathrm{O}(h^{-2}\cdot h^3) = \mathrm{O}(h).$$

We recall that the square of the L^2 norm of a basis function is $\mathrm{O}(h^3)$ while that of its curl is $\mathrm{O}(h)$. When we put to zero the degrees of freedom on the two faces adjacent to E, we obtain a vector $\mathbf{w}$ with a nonvanishing curl and therefore with a much larger energy:

$$\|\mathbf{w}\|^2_{H(\mathbf{curl};\Omega_i)} \sim \|\,\mathbf{curl}\,\mathbf{w}\|^2_{L^2(\Omega_i)^3} = \mathrm{O}(h^{-2}\cdot h) = \mathrm{O}(1/h).$$

A nonoverlapping domain decomposition algorithm which employs the standard three-dimensional edge element basis is expected to exhibit a condition number that grows at least as h^{-2}, which is the same growth exhibited by the original stiffness matrix K.

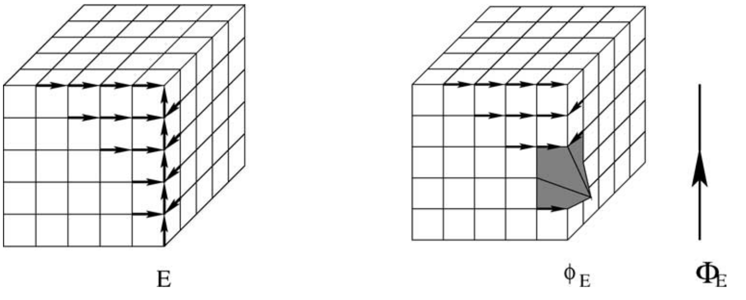

Fig. 1. Two types of basis functions associated to a subdomain edge: the standard basis (left) and one consisting of gradients of continuous, scalar, nodal functions associated to nodes internal to the coarse edges and one vector function with unitary tangential component along the coarse edges.

3. A change of basis

The discussion in the previous section hints that, before devising iterative substructuring algorithms for Problem (2), a change of basis may be necessary.

Definition 3.1 (New basis).

1. The basis functions associated with the edges inside Ω_i and with those in the interior of the faces are the same as for the standard basis;
2. the basis functions associated with a subdomain edge E are:
   (a) one vector function Φ_E with unitary tangential component along E and vanishing tangential component along all the fine edges that lie on the remaining coarse edges, on the faces, and in the interior of Ω_i;
   (b) the gradients of continuous, scalar, nodal functions associated to the interior nodes of E; these scalar functions take the value zero at all the nodes in the closure of Ω_i except at one node in the interior of E, where they are equal to one.

This new basis is described in Fig. 1. The new basis functions are introduced only for the coarse subdomain edges and the new degrees of freedom can still be partitioned into interior, face, and edge. The construction of this new basis does not rely on the fact that the substructures are elements of a coarse mesh or have a special shape.

4. Dual-primal FETI algorithms

We assume from now on that vectors of primal degrees of freedom are relative to the new basis introduced in the previous section. We define the intersection between the subdomain boundaries by Γ and note that it consists of faces F which are shared by exactly two subdomains, and subdomain edges and vertices, which are shared by more than two subdomains; see, e.g., Fig. 1. We then rewrite Eq. (2) as:

$$\widetilde{K}\tilde{u} + B^{\mathrm{T}}\lambda = \tilde{f}, \qquad B\tilde{u} = 0. \tag{3}$$

Here, the vector of degrees of freedom $\tilde{u}$ belongs to a larger finite element space $\widetilde{X}_h \supset X_h$, consisting of vectors that have a discontinuous tangential component along Γ; $\widetilde{K}$ is the stiffness matrix relative to the space $\widetilde{X}_h$. The second equation enforces the continuity of the tangential component along Γ and λ is the Lagrange multiplier associated to these constraints. See [3,1,7] for details.

An equation for λ can be obtained by eliminating the primal variable $\tilde{u}$. We obtain:

$$M^{-1}(F\lambda - d) = 0, \qquad F = B\widetilde{K}^{-1}B^{\mathrm{T}}. \tag{4}$$

Here, M^{-1} is a suitable symmetric, positive definite preconditioner. Once λ is found by an iterative method like Conjugate Gradient the solution $\tilde{u}$ can be found.

The form of the preconditioner M^{-1} is standard and can be found in, e.g., [3,4] or [7, Section 3]. We recall, in particular, that one application of M^{-1} requires the solution of a *Dirichlet* problem on each subdomain. In addition, suitable scaling matrices need to be applied, which are constructed with the coefficients a and b in (1); see next section for more details.

The matrix $\widetilde{K}$ (and therefore its inverse, required for the application of F) depends on the exact choice of the discontinuous space $\widetilde{X}_h$. In dual-primal FETI algorithms, a certain number of continuity constraints (*primal constraints*) are enforced for the vectors $\tilde{u} \in \widetilde{X}_h$. The application of $\widetilde{K}^{-1}$ to a vector requires the solution of *Neumann* problems on the subdomains and an additional *coarse*, *global* problem the size of which equals the number of primal constraints. The selection of primal constraints is of key importance in the development of dual-primal FETI algorithms; see [3,4]. The primal constraints are for our edge element approximations associated to the subdomain edges.

We first impose that the average (zero-th order moment) of the tangential component of vectors in $\widetilde{X}_h$ is continuous along every coarse edge E, independently of which substructure is used in the evaluation of this average. We note that these primal constraints are the coarse degrees of freedom associated to $\mathcal{T}_H$, in case the subdomain partition $\mathcal{T}_H$ coincides with a coarse mesh. As for similar algorithms for three-dimensional, continuous nodal elements, these primal constraints are not sufficient to ensure good condition number bounds; see [3,4]. We impose that also the *first order moments* of the tangential component are continuous along every coarse edge E. This choice therefore gives two primal constraints associated to a subdomain edge, and therefore a coarse problem of small size.

5. Theoretical and numerical results

In order to obtain bounds for the condition number of the FETI operator $M^{-1}F$, we need to make further assumptions on the coefficients. We first assume that a is uniformly bounded from zero. This assumption is required only for the theory but in practice good convergence is also obtained for a small or zero. Furthermore, while similar algorithms for two-dimensional approximations can be made robust with respect to the jumps of both coefficients in a straightforward way (see [7] and the references therein), this is not the case in three dimensions. We assume therefore that only one of the coefficients a and b may have arbitrarily large jumps across the substructures Ω_i, while the other is assumed to be continuous across the substructures or to have small jumps. The scaling matrices used for the construction of the preconditioner M^{-1} use the coefficient that has jumps. We have the following result:

Theorem 5.1. *Let a be uniformly bounded from below. There is then a constant independent of h, the number of subdomains, or the jumps of one of the coefficients, such that the condition number bound*

$$\kappa(M^{-1}F) \leqslant C\big(1 + \log(H/h)\big)^q,$$

holds with $q = 4$.

The proof relies on decomposition results for the tangential traces of edge element vectors in the new basis associated to the subdomain edges and faces.

We now present some simple numerical tests. We consider the domain $\Omega = (0,1)^3$ and uniform triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$. The coarse triangulation $\mathcal{T}_H$ consists of N^3 cubical elements, with $H = 1/N$. The fine one $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$ and consists of n^3 cubical elements, with $h = 1/n$. The substructures Ω_i are chosen as the elements of $\mathcal{T}_H$. We assume $a = b = 1$. Table 1, left, shows that the condition numbers of the preconditioned FETI operator $M^{-1}F$ are small and bounded independently of $h = 1/n$: their growth with H/h is consistent with $q = 2$

Table 1
Condition number of $M^{-1}F$ versus H/h and n, with (left) and without (right) a change of basis

H/h	8	6	4	3	2	H/h	8	6	4	3	2
$n=8$	–	–	2.213	–	1.869	$n=8$	–	–	151.6	–	1.869
$n=16$	3.076	–	2.742	–	1.936	$n=16$	643.7	–	607.6	–	1.936
$n=24$	3.566	3.322	2.838	2.48	1.960	$n=24$	1449	1429	1365	1429	1.960
$n=32$	3.774	–	2.899	–	1.969	$n=32$	2576	–	2427	–	1.969
$n=40$	3.866	–	2.926	–	–	$n=40$	4024	–	3793	–	–
$n=48$	3.951	3.537	2.944	–	–	$n=48$	5795	5712	5462	–	–

in Theorem 5.1. In Table 1, right, we show the condition number of the same algorithm but without performing the change of basis of Section 3. The condition numbers are very high and are consistent with a quadratic growth in n, thus confirming our analysis in Section 2. We note that the algorithms relative to the last column ($H/h=2$) are special cases for which the tangential component is continuous along the subdomain edges and for which a change of basis is not necessary.

References

[1] M. Bhardwaj, K. Pierson, G. Reese, T. Walsh, D. Day, K. Alvin, J. Peery, C. Farhat, M. Lesoinne, Salinas: A scalable software for high-performance structural and solid mechanics simulations, in: Proceedings of the IEEE/ACM SC2002 Conference, Baltimore, MD, November 16–22, 2002.

[2] M. Dryja, B.F. Smith, O.B. Widlund, Schwarz analysis of iterative substructuring algorithms for elliptic problems in three dimensions, SIAM J. Numer. Anal. 31 (6) (1994) 1662–1694.

[3] C. Farhat, M. Lesoinne, P. LeTallec, K. Pierson, D. Rixen, FETI-DP: a dual-primal unified FETI method. I. A faster alternative to the two-level FETI method, Int. J. Numer. Methods Engrg. 50 (7) (2001) 1523–1544.

[4] A. Klawonn, O.B. Widlund, M. Dryja, Dual-primal FETI methods for three-dimensional elliptic problems with heterogeneous coefficients, SIAM J. Numer. Anal. 40 (1) (2002) 159–179.

[5] P. Monk, Finite Element Methods for Maxwell's Equations, Numerical Mathematics and Scientific Computation, The Clarendon Press, Oxford University Press, New York, 2003.

[6] J.-C. Nédélec, Mixed finite elements in R^3, Numer. Math. 35 (1980) 315–341.

[7] A. Toselli, X. Vasseur, Dual-primal FETI algorithms for edge element approximations: two-dimensional h and p finite elements on shape-regular meshes, Tech. Report 04-01, Seminar for Applied Mathematics, ETH Zürich, 2004, SIAM J. Numer. Anal., in press.