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Probability Theory

Dynamical evaluations

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Abstract

In this Note we announce the following result: under a domination condition, an $\mathcal{F}_t$-consistent evaluation is an $\mathcal{E}^g$-evaluation. ***To cite this article: S. Peng, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Évaluations dynamiques. Dans cette Note nous annonçons le résultat suivant : sous une hypothèse de domination, une évaluation $\mathcal{F}_t$-consistante est une $\mathcal{E}^g$-évaluation. ***Pour citer cet article : S. Peng, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Soient $\{\mathcal{F}_t\}_{t\geqslant 0}$ une filtration engendrée par un mouvement brownien d-dimensionnel défini sur un espace probabilisé $(\Omega, \mathcal{F}, P)$, $L^2(\mathcal{F}_t)$ l'ensemble des variables aléatoires $\mathcal{F}_t$-mesurables avec $E[|X|^2] < \infty$, $L^2_{\mathcal{F}}(0,T)$ l'ensemble des processus $\mathcal{F}_t$-adaptés avec $E[\int_0^T |\phi_t|^2\,\mathrm{d}t] < \infty$. On fixe un intervalle $[0, T]$.

Définition 0.1. Un système d'opérateurs :

$$\mathcal{E}_{s,t}[X]: X \in L^2(\mathcal{F}_t) \to L^2(\mathcal{F}_s), \quad 0 \leqslant s \leqslant t \leqslant T,$$

est une évaluation $\mathcal{F}_t$-consistante définie sur $[0, T]$ s'il vérifie les hypothèses suivantes : pour chaque r, s et t dans $[0, T]$ tels que $0 \leqslant r \leqslant s \leqslant t \leqslant T$ et pour chaque X, $X' \in L^2(\mathcal{F}_t)$, on a

(A1) $\mathcal{E}_{s,t}[X] \geqslant \mathcal{E}_{s,t}[X']$, p.s., si $X \geqslant X'$, p.s. ;

(A2) $\mathcal{E}_{t,t}[X] = X$, p.s. ;

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(A3) $\mathcal{E}_{r,s}[\mathcal{E}_{s,t}[X]] = \mathcal{E}_{r,t}[X]$, p.s. ;
(A4) $1_A\mathcal{E}_{s,t}[X] = 1_A\mathcal{E}_{s,t}[1_A X]$, p.s. $\forall A \in \mathcal{F}_s$.
Une hypothèse supplémentaire est :
(A4$_0$) $\mathcal{E}_{s,t}[0] = 0$, p.s.
Soit $g = g(\omega, t, y, z)$ une fonction réelle définie sur $\Omega \times [0, T] \times R \times R^d$ qui vérifie (3). Pour chaque $t \in [0, T]$ et $X \in L^2(\mathcal{F}_t)$, soit $(Y_s, Z_s)_{0\leqslant s\leqslant t}$ la solution de l'EDSR (2). On définit $\mathcal{E}^g_{s,t}[X] := Y_s$, $s \in [0, t]$.

Théorème 0.2. *Si la fonction g vérifie* (i) *et* (ii) *de* (3)*, alors le système d'opérateurs* $\{\mathcal{E}^g_{s,t}[\cdot]\}_{0\leqslant s\leqslant t\leqslant T}$ *est une évaluation $\mathcal{F}_t$-consistante définie sur* $[0, T]$*. De plus, $\mathcal{E}^g_{s,t}[\cdot]$ est dominé par $\mathcal{E}^{g_\mu}_{s,t}[\cdot]$, avec $g_\mu(y, z) := \mu(|y| + |z|)$, au sens suivant : pour chaque $s, t \in [0, T]$, tels que $s \leqslant t$,*

$$\mathcal{E}^g_{s,t}[X] - \mathcal{E}^g_{s,t}[Y] \leqslant \mathcal{E}^{g_\mu}_{s,t}[X - Y], \quad \textit{p.s.}, \ \forall X, Y \in L^2(\mathcal{F}_t).$$

Si, de plus, $g(s, 0, 0) \equiv 0$, alors (A4$_0$) *est vérifié : $\mathcal{E}^g_{s,t}[0] \equiv 0$.*

Dans la fonction g_μ, μ est la constante de Lipschitz de g dans (3)(ii).

Théorème 0.3. *Soit $\{\mathcal{E}_{s,t}[\cdot]\}_{0\leqslant s\leqslant t\leqslant T}$ une évaluation $\mathcal{F}_t$-consistante définie sur* $[0, T]$ *dominée par $\mathcal{E}^{g_\mu}[\cdot]$ au sens suivant : pour chaque $s, t \in [0, T]$, tels que $s \leqslant t$,*

$$\mathcal{E}_{s,t}[X] - \mathcal{E}_{s,t}[Y] \leqslant \mathcal{E}^{g_\mu}_{s,t}[X - Y], \quad \textit{p.s.}, \ \forall X, Y \in L^2(\mathcal{F}_t).$$

Si de plus (A4$_0$) *est vérifiée, i.e., $\mathcal{E}_{s,t}[0] \equiv 0$, alors il existe une fonction réelle g définie sur $\Omega \times [0, T] \times R \times R^d$ vérifiant* (3) *et $g(\cdot, 0, 0) \equiv 0$ telle que pour chaque $s, t \in [0, T]$, tels que $s \leqslant t$, on a*

$$\mathcal{E}^g_{s,t}[X] = \mathcal{E}_{s,t}[X], \quad \textit{p.s.}\ \forall X \in L^2(\mathcal{F}_t).$$

On peut aussi affaiblir l'hypothèse (A4$_0$).

1. Introduction

Let $(\Omega, \mathcal{F})$ be a measurable space and let $\{\mathcal{F}_t\}_{t\geqslant 0}$ be a filtration defined in this space. In this paper $\mathcal{F}_t$ represents the information acquired by an economic agent (an individual, a firm, or even a market) during the period $[0, t]$. We denote the set of all $\mathcal{F}_t$-measurable real valued random variables by $m\mathcal{F}_t$. Assume that at the present time s, this agent evaluates a future risky payoff (one may think of a contingent claim) X, with maturity $t \geqslant s$. This X is an $\mathcal{F}_t$-measurable random variable. When the time t comes, the agent will know the money based value of X. We denote this evaluated value by $\mathcal{E}_{s,t}[X]$. It is reasonable to assume that $\mathcal{E}_{s,t}[X] \in m\mathcal{F}_s$. $\{\mathcal{E}_{s,t}[\cdot]\}_{0\leqslant s\leqslant t<\infty}$ forms a system of operators:

$$\mathcal{E}_{s,t}[\cdot] : m\mathcal{F}_t \to m\mathcal{F}_s, \quad 0 \leqslant s \leqslant t < +\infty. \tag{1}$$

We will make the following axiomatic assumptions: for each $0 \leqslant r \leqslant s \leqslant t$ and for each X, $X' \in m\mathcal{F}_t$,

(A1) $\mathcal{E}_{s,t}[X] \geqslant \mathcal{E}_{s,t}[X']$, if $X \geqslant X'$;
(A2) $\mathcal{E}_{t,t}[X] = X$;
(A3) $\mathcal{E}_{r,s}[\mathcal{E}_{s,t}[X]] = \mathcal{E}_{r,t}[X]$;
(A4) $1_A\mathcal{E}_{s,t}[X] = 1_A\mathcal{E}_{s,t}[1_A X]$, $\forall A \in \mathcal{F}_s$.

A system of operators (1) satisfying (A1)–(A4) is called an $\mathcal{F}_t$-consistent evaluation.

Remark 1. The meaning of (A1) and (A2) are obvious. Condition (A3) means that at the time $r \leqslant s$, the value $\mathcal{E}_{s,t}[X]$ is also regarded as a risky payoff with the maturity s. The price of this risky payoff $\mathcal{E}_{r,s}[\mathcal{E}_{s,t}[X]]$ is the same as the price of the original derivative X with maturity t, i.e., $\mathcal{E}_{r,t}[X]$.

Remark 2. In (A4) 1_A is the indicator of $A \in \mathcal{F}_s$, i.e., $1_A(\omega) = 1$, if $\omega \in A$, $1_A(\omega) = 0$, otherwise. 1_A is considered as a "digital option". (A4) means that, at time s, the agent knows whether 1_A worths 1 or zero. If it worths 1, then the value $\mathcal{E}_{s,t}[1_A X]$ is the same as $\mathcal{E}_{s,t}[X]$ since the two outcomes X and $1_A X$ are exactly the same. Note that this condition does not indicate the case where 1_A worths zero. If we assume an additional condition:

(A4$_0$) $\mathcal{E}_{s,t}[0] \equiv 0$,

then $\mathcal{E}_{s,t}[1_A X] = 0$ when $1_A = 0$. We note that (A4) + (A4$_0$) is equivalent to

(A4′) $1_A \mathcal{E}_{s,t}[X] = \mathcal{E}_{s,t}[1_A X]$, $\forall A \in \mathcal{F}_s$.

2. $\mathcal{F}_t$-consistent evaluation under Brownian filtration

Our framework will be a given probability space $(\Omega, \mathcal{F}, P)$ where $\{\mathcal{F}_t\}_{0 \leqslant t < \infty}$ is the filtration generated by a d-dimensional Brownian motion $(B_t)_{t \geqslant 0}$ defined on $(\Omega, \mathcal{F}, P)$. For each time $t \geqslant 0$, we denote by $L^2(\mathcal{F}_t)$ a subspace of $m\mathcal{F}_t$ such that $E[X^2] < \infty$. Without loss of generality, we will work within a given interval $[0, T]$.

Definition 2.1. A system of operators:

$$\mathcal{E}_{s,t}[X] : X \in L^2(\mathcal{F}_t) \to L^2(\mathcal{F}_s), \quad 0 \leqslant s \leqslant t \leqslant T,$$

is called an $\mathcal{F}_t$-consistent evaluation defined on $[0, T]$ if it satisfies (A1)–(A4), where the relations '$=$' and '$\geqslant$' are in the sense of 'P-a.s.'.

3. $\mathcal{E}^g$-evaluations induced by BSDE

We denote by $L^2_{\mathcal{F}}(0, t; R^m)$ the set of all R^m-valued and $(\mathcal{F}_t)_{t \geqslant 0}$-adapted stochastic processes such that $E \int_0^t |\phi_s|^2 \, \mathrm{d}s < \infty$ and by $S^2_{\mathcal{F}}(0, t)$ the processes in $L^2_{\mathcal{F}}(0, t) = L^2_{\mathcal{F}}(0, t; R)$ with continuous paths such that $E[\sup_{0 \leqslant s \leqslant t} |\phi_s|^2] < \infty$. For each $t \in [0, T]$, we consider the following backward stochastic differential equation with terminal time t:

$$Y_s = X + \int_s^t g(r, Y_r, Z_r) \, \mathrm{d}r - \int_s^t Z_r \, \mathrm{d}B_r, \quad s \in [0, t]. \tag{2}$$

Here the function $g : (\omega, t, y, z) \in \Omega \times [0, T] \times R \times R^d \to R$ satisfies the following usual assumptions of BSDE: for each $\forall y, y' \in R$ and $z, z' \in R^d$,

$$\begin{cases} \text{(i)} & g(\cdot, y, z) \in L^2_{\mathcal{F}}(0, T); \\ \text{(ii)} & \big|g(t, y, z) - g(t, y', z')\big| \leqslant \mu\big(|y - y'| + |z - z'|\big). \end{cases} \tag{3}$$

Assumption (3)(ii) is called the Lipschitz condition of g w.r.t. (y, z). It is equivalent to: g is dominated by $g_\mu(y, z) := \mu(|y| + |z|)$ in the following sense:

$$g(t, y, z) - g(t, y', z') \leqslant g_\mu(y - y', z - z'), \quad \forall y, y' \in R, \ z, z' \in R^d.$$

It is known that for each given $X \in L^2(\mathcal{F}_t)$, there exists a unique solution $(Y, Z) \in S^2_{\mathcal{F}}(0, t) \times L^2_{\mathcal{F}}(0, t; R^d)$ of BSDE (2). We denote $\mathcal{E}^g_{s,t}[X] := Y_s$. We thus define a system of operators:

$$\mathcal{E}^g_{s,t}[X]: X \in L^2(\mathcal{F}_t) \to L^2(\mathcal{F}_s), \quad 0 \leqslant s \leqslant t \leqslant T.$$

This system is completely determined by the above given function g. We have

Theorem 3.1. *We assume that the function g satisfies* (i) *and* (ii) *of* (3). *Then the system of operators $\{\mathcal{E}^g_{s,t}[\cdot]\}_{0\leqslant s\leqslant t\leqslant T}$ is an $\mathcal{F}_t$-consistent evaluation defined on $[0,T]$. Moreover $\mathcal{E}^g_{s,t}[\cdot]$ is dominated by $\mathcal{E}^{g_\mu}_{s,t}[\cdot]$ in the following sense: for each $s,t \in [0,T]$, such that $s \leqslant t$,*

$$\mathcal{E}^g_{s,t}[X] - \mathcal{E}^g_{s,t}[Y] \leqslant \mathcal{E}^{g_\mu}_{s,t}[X-Y], \quad \text{a.s.}, \ \forall X, Y \in L^2(\mathcal{F}_t). \tag{4}$$

Furthermore, if $g(s,0,0) \equiv 0$, then (A4$_0$) *holds: $\mathcal{E}^g_{s,t}[0] \equiv 0$.*

4. $\mathcal{F}_t$-consistent evaluation determined by a function g

A more interesting problem is that if a given $\mathcal{F}_t$-consistent evaluation $\mathcal{E}[\cdot]$ is dominated $\mathcal{E}^{g_\mu}[\cdot]$ for a sufficiently large μ, can we find a function g such that $\mathcal{E}[\cdot]$ coincides with $\mathcal{E}^g[\cdot]$? We announce the following result:

Theorem 4.1. *We assume that an $\mathcal{F}_t$-consistent evaluation $\{\mathcal{E}_{s,t}[\cdot]\}_{0\leqslant s\leqslant t\leqslant T}$ defined on $[0,T]$ is dominated by $\mathcal{E}^{g_\mu}[\cdot]$ in the following sense: for each $s,t \in [0,T]$, such that $s \leqslant t$,*

$$\mathcal{E}_{s,t}[X] - \mathcal{E}_{s,t}[Y] \leqslant \mathcal{E}^{g_\mu}_{s,t}[X-Y], \quad \text{a.s.}, \ \forall X, Y \in L^2(\mathcal{F}_t). \tag{5}$$

We also assume that (A4$_0$) *hold. Then there exists a function $g : \Omega \times [0,T] \times R \times R^d \to R$ satisfying* (3) *and $g(\cdot,0,0) \equiv 0$ such that, for each $s,t \in [0,T]$ with $s \leqslant t$, we have*

$$\mathcal{E}^g_{s,t}[X] = \mathcal{E}_{s,t}[X], \quad \text{a.s.}, \ \forall X \in L^2(\mathcal{F}_t). \tag{6}$$

In the situation where $\mathcal{E}[\cdot]$ does not satisfy (A4$_0$), we have

Corollary 4.2. *We assume that an $\mathcal{F}_t$-consistent evaluation $\{\mathcal{E}_{s,t}[\cdot]\}_{0\leqslant s\leqslant t\leqslant T}$ defined on $[0,T]$ is dominated by $\mathcal{E}^{g_\mu}_{s,t}[\cdot]$ in the sense of* (5) *and*

$$\mathcal{E}^{-g_\mu+g^0}_{s,t}[0] \leqslant \mathcal{E}_{s,t}[0] \leqslant \mathcal{E}^{g_\mu+g^0}_{s,t}[0], \quad \text{a.s.}, \ 0 \leqslant s < t \leqslant T,$$

for a given $g^0(\cdot) \in L^2_{\mathcal{F}}(0,T)$. Then there exists a function $g : \Omega \times [0,T] \times R \times R^d \to R$ satisfying (3) *and $g(t,0,0) \equiv g^0(t)$ such that, for each $s,t \in [0,T]$ with $s \leqslant t$, we have* (6).

We can still treat a more general situation. Let $K^0 \in L^2_{\mathcal{F}}(0,T)$ be given such that its paths are a.s. RCLL and such that $E[\sup_{0\leqslant t\leqslant T} |K^0_t|^2] < \infty$. We consider the following BSDE

$$Y_s = X + K^0_t - K^0_s + \int_s^t g(r, Y_r, Z_r)\,\mathrm{d}r - \int_s^t Z_r\,\mathrm{d}B_r, \quad s \in [0,t]. \tag{7}$$

It is easy to check, by changing variable $\overline{Y_s} := Y_s + K^0_s$, that this BSDE has a unique solution $(Y,Z) \in L^2_{\mathcal{F}}(0,T; R^{1+d})$ such that $Y + K^0 \in S^2_{\mathcal{F}}(0,T)$. We denote $\mathcal{E}^g_{s,t}[X; K^0] := Y_s$, $s \in [0,t]$. In finance, an increasing process K^0 represents an accumulated dividend. The interesting thing here is that, the system of operators

$$\mathcal{E}^g_{s,t}[\cdot; K^0] : L^2(\mathcal{F}_t) \to L^2(\mathcal{F}_s), \ 0 \leqslant s \leqslant t \leqslant T,$$

is still an $\mathcal{F}_t$-consistent evaluation. In fact we have

Theorem 4.3. *Let K^0 be given as the above. We assume that the function g satisfies* (i) *and* (ii) *of* (3) *and that* $g(s,0,0)\equiv 0$. *Then the system of operators of* $\{\mathcal{E}^g_{s,t}[\cdot;K^0]\}_{0\leqslant s\leqslant t\leqslant T}$ *is an $\mathcal{F}_t$-consistent evaluation defined on* $[0,T]$. *It is dominated by* $\mathcal{E}^{g_\mu}_{s,t}[\cdot]$ *in the following sense*: *for each* $s,t\in[0,T]$, *such that* $s\leqslant t$,

$$\mathcal{E}^g_{s,t}[X;K^0]-\mathcal{E}^g_{s,t}[Y;K^0]\leqslant \mathcal{E}^{g_\mu}_{s,t}[X-Y],\quad \textit{a.s.},\ \forall X,Y\in L^2(\mathcal{F}_t).$$

Moreover,

$$\mathcal{E}^{-g_\mu}_{s,t}[0;K^0]\leqslant \mathcal{E}^g_{s,t}[0;K^0]\leqslant \mathcal{E}^{g_\mu}_{s,t}[0;K^0].$$

To this theorem we can still add its converse result:

Corollary 4.4. *We assume that an $\mathcal{F}_t$-consistent evaluation* $\{\mathcal{E}_{s,t}[\cdot]\}_{0\leqslant s\leqslant t\leqslant T}$ *defined on* $[0,T]$ *is dominated by* $\mathcal{E}^{g_\mu}_{s,t}[\cdot]$ *in the sense of* (5) *and that, there exist an RCLL process* $K^0\in L^2_{\mathcal{F}}(0,T)$ *with* $E[\sup_{0\leqslant t\leqslant T}|K^0_t|^2]<\infty$, *such that, for each* $0\leqslant s\leqslant t\leqslant T$,

$$\mathcal{E}^{-g_\mu}_{s,t}[0;K^0]\leqslant \mathcal{E}_{s,t}[0]\leqslant \mathcal{E}^{g_\mu}_{s,t}[0;K^0].$$

Then there exists a function $g:\Omega\times[0,T]\times R\times R^d\to R$ *satisfying* (3) *and* $g(\cdot,0,0)\equiv 0$ *such that, for each* $s,t\in[0,T]$ *with* $s\leqslant t$, *we have*

$$\mathcal{E}_{s,t}[X]=\mathcal{E}^g_{s,t}[X;K^0],\quad \textit{a.s.}\ \forall X\in L^2(\mathcal{F}_t). \tag{8}$$

All the proofs of the above results can be found in our recent paper [5]. For example Theorems 4.3 and 3.1 are given as Proposition 2.11 and Corollary 2.9 in [5]. Theorem 4.1, Corollary 4.2 and Corollary 4.4 are given in Theorem 3.1, Corollary 3.3 and Corollary 3.2 in [5]. Theorem 4.1 has nontrivially generalized the result in [1] for $\mathcal{F}_t$-consistent expectations. The notion of g-expectation, firstly introduced in [3], is a special case of g-evalutation. For details, see [2,4–6] and [7].

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